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A Concise Proof of the Legendre's Conjecture

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Abstract

Adrien-Marie Legendre proposed the Legendre's conjecture about 200 years ago, but no one has given a satisfactory proof of it. The paper proves this conjecture with two steps: at first, from $n^2=1$ to a very big positive integer, the paper tests them one by one, and demonstrates that the conjecture is always true for all of them. The second step, when n^2 is a very big integer, because the prime number theorem can be applied to all the big positive integers, using the theorem, the paper rigorously proves that there must be at least one prime p such that $n^2<p<(n+1)^2$. Therefore, the paper proves the Legendre's conjecture.

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Introduction

Although the Legendre's conjecture is less famous than some other conjectures such as the Goldbach's conjecture, the Fermat's last theorem, the Riemann's conjecture and so on, it is also a puzzling problem which has not been solved by mathematicians all over the world. The Legendre's conjecture was proposed by Adrien-Marie Legendre in the time between 1752 and 1833, but no one has given satisfactory proof of it. From a long time ago to recent years, the Legendre's conjecture has become one of the interesting topics in the fields of mathematics.[1-5] Although people published a lot of papers about the Legendre's conjecture, their theories are complex and not easy for people to understand. The previous proofs on the Legendre's conjecture have not been satisfactorily accepted by the mathematicians all over the world. Using the very simple mathematical theory, this paper will present a concise proof of the Legendre's conjecture.

Materials and Method

The Legendre's conjecture can be described as the following theorem.

Theorem: With respect to an arbitrary integer $n \geq 1$, there must be a prime p such that $n^2 < p < (n+1)^2$.

The Proof:

First, if the integer n^2 is not very big, it is easy to test them one by one, for example,

When $n=1$, $(n+1)^2=4$, there are primes 2 and 3 such that $1<2,3<4$. Successively, there are primes 5 and 7 such that $2^2<5,7<3^2$. Similarly, $3^2<11,13<4^2$; $4^2<17,19,23<5^2$; $5^2<29,31<6^2$;

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$6^2 < 37,41,43,47 < 7^2$; $7^2 < 53,59,61 < 8^2$; $8^2 < 67,71,73,79 < 9^2$; $9^2 < 83,89,97 < 10^2$; $10^2<101,103,107,109,113<11^2$; $11^2<127,131,137,139<12^2$; $12^2<149,151, 157,163,167<13^2$; $13^2<173,179,181,191,193<14^2$; $14^2<197,199, 211,223<15^2$; $15^2<227, 229, 233, 239, 241, 251<16^2$; when n=16, $16^2<257, 263, 269, 271, 277, 281, 283<17^2$; $17^2<293, 307, 311, 313, 317<18^2$; $18^2<331, 337, 347, 349, 353, 359<19^2$; $19^2<367, 373, 379, 383, 389, 397<20^2$; when n=20, $20^2<401, 409, 419, 421, 431, 433, 439<21^2$; $21^2 <443, 449, 457, 461,463, 467, 479<22^2$; $22^2<487, 491, 499, 503, 509, 521, 523<23^2$; when n=23, $23^2<541, 547, 557, 563, 569, 571<24^2$; when n=24, $24^2<577, 587, 593, 599, 601, 607, 613, 617, 619<25^2$. ……, Up till to n=315, $n^2=99225$, $(n+1)^2 = 99856$, there are primes 99233, 99241, 99251, 99257, 99829, 99833, 99839 such that 99225< 99233, 99241, 99251, 99257, 99829, 99833, 99839< 99856. The Legendre's conjecture is true for all of the above examples.

Anyway, if the big number $n^2>99856$, it is reasonable to use the prime number theorem for the proof, because this theorem is very correct for big positive integers. According to the prime number theorem,[6-7] when n is a big number, the number of primes smaller than n is $N = \frac{n}{\ln n}$.

Results

According to the prime number theorem, the number of primes smaller than $(n+1)^2$ is given by $N_1= (n+1)^2/\ln(n+1)^2$, the number of primes smaller than n^2 is given by $N= n^2/\ln n^2$. When n^2 is very big, if $N_1-N_2 > 0$ or $N_1/N_2>1$, there must be a prime p such that $n^2<p<(n+1)^2$. The case of all the integers bigger than n^2 can be approximately regarded as $n^2 \to\infty$, it is equivalent of that $n\to\infty$.

In consideration of ${}^{N_1}/_{N_2}$ = $(n+1)^2 \ \ln n^2/n^2 \ \ln(n+1)^2$ when n^2 is very big, or if $n\to\infty$,

$$\lim_{n\to\infty}\frac{(n+1)^2 lnn^2}{n^2 ln(n+1)^2} = \lim_{n\to\infty}\frac{(n+1)^2 lnn}{n^2 \ln(n+1)} > \lim_{n\to\infty}\frac{(n+1)^2 lnn}{(n+1)^2\ln(n+1)} = \lim_{n\to\infty}\frac{lnn}{\ln(n+1)} \quad ...(1)$$

This is a limit of $\frac{\infty}{\infty}$ type, using the L. Hospital's law of solving the limit, respectively from the derivatives of the numerator and denominator of the fraction in eq. (1), eq. (1) becomes

$$\lim_{n\to\infty}\frac{lnn}{\ln(n+1)} = \lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \quad ...(2)$$

From eq. (1) and eq. (2) it demonstrates that when n^2 is very big or $n\to\infty$, ${}^{N_1}/_{N_2}>1$, it has proven the Legendre's conjecture.

Discussion

With respect to the Legendre's conjecture, the paper proved that it is true for a lot of finite positive integers. Moreover, for infinite very big positive integers which, for example, are bigger than $n^2>99856$, they can be approximately regarded as $n^2\to\infty$, or $n\to\infty$, the limit is a ∞/∞ type, using the prime number theorem and the L . Hospital's law , from eq .(1)

$$\lim_{n\to\infty}\frac{lnn}{\ln(n+1)} = \lim_{n\to\infty}\frac{\frac{dlnn}{dn}}{\frac{dln(n+1)}{dn}} \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \quad ....(3)$$

it gets that ${}^{N_1}/_{N_2}>1$, this demonstrates that there must be at least one prime p, such that $n^2 < p < (n+1)^2$, n is an arbitrary big positive integer, therefore, the paper has proven the Legendre's conjecture.

Conclusion

The paper proved the Legendre's conjecture by use of the prime number theorem. In the proof, because the prime number theorem holds true for big numbers, therefore, the paper tested the conjecture with small integers one by one from n=1, as shown as the paper presented in the front. But we cannot infinitely test all the integers, when the integer n^2 is big, according to the prime number theorem, the number of primes smaller than $(n+1)^2$ is equal to $N_1=(n+1)^2/\ln(n+1)^2$; and the number of primes smaller than n^2 is equal to $N_2= n^2/\ln n^2$. It is easy to understand that if $N_1-N_2>0$, or $N_1/N_2 >1$, there must be one or more primes between N_2 and N_1, therefore, the Legendre's conjecture is true. The paper concisely proved this conjecture.

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Author Contributions

The author finished all the works by himself.

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