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Wijsman Rough λ Statistical Convergence of Order α of Triple Sequence of Functions

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ABSTRACT

In this paper, using the concept of natural density, we introduce the notion of Wijsman rough λ statistical convergence of order α triple sequence of functions. We define the set of Wijsman rough λ statistical convergence of order α of limit points of a triple sequence spaces of functions and obtain Wijsman λ statistical convergence of order α criteria associated with this set. Later, we prove that this set is closed and convex and also examine the relations between the set of Wijsman rough λ statistical convergence of order α of cluster points and the set of Wijsman rough λ statistical convergence of order α limit points of a triple sequences of functions.

Keywords: Wijsman rough λ^{α} statistical convergence, Natural density, triple sequences of functions, order α.

INTRODUCTION

The idea of statistical convergence was introduced by Steinhaus[17] and also independently by Fast [2] for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence.

Let K be a subset of the set of positive integers $N\times N\times N$, and let us denote the set $\{(m,n,k)\in K:m\leq u,n\leq v,k\leq w\}$ by K_uvw. Then the natural density of K is given by

$$\delta(K) = lim_{uvw\to\infty} \frac{|K_{uvw}|}{uvw},$$

where $|K_{uvw}|$ denotes the number of elements in K_{uvw}. Clearly, a finite subset has natural density zero, and we have $\delta(K^c)=1-\delta(K)$ where $K^c=N\backslash K$ is the complement of K. If $K_1\subseteq K_2$, then $\delta(K_1)\leq\delta(K_2)$.

Throughout the paper, R denotes the real of three dimensional space with metric (X,d). Consider a triple sequence x=(x_mnk) such that $x_{mnk}\in R, m,n,k\in N$.

A triple sequence $x=(x_{mnk})$ is said to be statistically convergent to $0\in R$, written as st-limx=0, provided that the set

$$\{(m,n,k)\in N_3:|x\_mnk,0|\geq\delta\}$$

has natural density zero for any $\varepsilon>0$. In this case, 0 is called the statistical limit of the triple sequence x.

If a triple sequence is statistically convergent, then for every $\varepsilon>0$, infinitely many terms of the sequence may remain outside the ε- neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence $x=(x_{mnk})$ satisfies some property P for all m,n,k except a set of natural density zero, then we say that the triple sequence x satisfies P for almost all (m,n,k) and we abbreviate this by a.a. (m,n,k).

Let $(x_{minjkl}))$ be a sub sequence of $x=(x_{mnk})$. If the natural density of the set $K=\{(m_i,n_j,k_l)\in N_3:(i,j,l)\in N^3\}$ is different from zero, then (x_{minjkl}) is called a non thin sub sequence of a triple sequence x.

$c\in R$ is called a statistical cluster point of a triple sequence $x=(x_{mnk})$ provided that the natural density of the set

$$\{(m,n,k)\in N^3:|x_{mnk}-c|<\varepsilon\}$$

is different from zero for every $\varepsilon>0$. We denote the set of all statistical cluster points of the sequence x by Γ_x.

A triple sequence $x=(x_{mnk})$ is said to be statistically analytic if there exists a positive number M such that

$$\delta(\{(m,n,k)\in N^3:|x\_mnk|^{1/m+n+k}\geq M\})=0$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu[8], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar[1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal *et al.* [7] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

Let (X,ρ) be a metric space. For any non empty closed subsets $f,f_{mnk}\subset X(m,n,k\in I_{rst})$, we say that the triple sequence of functions of (f_{mnk}) is wijsman λ statistical convergent of order α to f is the triple sequence of functions $(d(f_{mnk},x))$ is statistically convergent to d(f,x), i.e., for $\varepsilon>0$ and for each $f\in X$

$$\lim_{rst} 1/(\lambda^{\alpha}_{rst}) |\{(m,n,k)\in I_{rst}:|d(f_{mnk},x)-d(f,x)|\geq\varepsilon\}|=0$$

In this case, we write $St\text{-}\lim_{mnk} f_{mnk}=f$ or $f_{mnk}\to f(WS)$. The triple sequence of functions of (f_{mnk}) is bounded if $\sup_{mnk} d(f_{mnk},x)<\infty$ for each $f\in X$.

In this paper, we introduce the notion of Wijsman rough statistical convergence of order α of triple sequence of functions. Defining the set of Wijsman rough $r\lambda^{\alpha}$ statistical convergence of order α limit points of a triple sequence of functions, we obtain to Wijsman $r\lambda^{\alpha}$ statistical convergence of order α criteria associated with this set. Later, we prove that this set of Wijsman $r\lambda^{\alpha}$ statistical convergence of order α of cluster points and the set of Wijsman rough $r\lambda^{\alpha}$ statistical convergence of order α limit points of a triple sequence of functions .

The α- density of a subet E of N. Let α be a real number such that $0<\alpha\leq 1$. The α- density of a subet E of N is defined by

$$\delta^{\alpha}(E)=\lim_{rst} 1/(rst)^{\alpha} |\{m\leq r,n\leq s,k\leq t:(m,n,k)\in E\}|$$

provided the limit exists, where $|\{m\leq r,n\leq s,k\leq t:(m,n,k)\in E\}|$ denotes the number of elements of E not exceeding (rst).

It is clear that any finite subset of N has a zero α density and $\delta^{\alpha}(E^c)=1-\delta^{\alpha}(E)$ does not hold for $0<\alpha<1$ in general, the equality holds only if $\alpha=1$. Note that the α- density of any set reduces to the natural density of the set in case $\alpha=1$.

A triple sequence (real or complex) can be defined as a function $x:N\times N\times N\to R(C)$, where N,R and C denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner *et al.* [9,10], Esi *et al.* [2-4], Datta *et al.* [5]Subramanian *et al.*[11], Debnath *et al.* [6], Esi *et al.*[12], and many others.

Throughout the paper let r be a nonnegative real number.

Definitions and Preliminaries

Definition

A triple sequence of functions (f_{mnk}) and $\alpha\in(0,1]$ said to be Wijsman $r\lambda^\alpha$- convergent to the function f denoted by $f_{mnk} \to^{r\lambda\alpha} f$, provided that $\forall\varepsilon>0$ $\exists(m\varepsilon,n_\varepsilon,k_\varepsilon)\in N^3: m\geq m_\varepsilon, n\geq n_\varepsilon, k\geq k_\varepsilon \Rightarrow \lim_{rst} 1/(\lambda^\alpha_{rst})$ $|\{(m,n,k)\in I_{rst}: |d(f_{mnk},x)-d(f,x)| < r+\varepsilon\}|=0$

The set

$$LIM^{r\lambda^\alpha} = \{f \in \mathbb{R}^3 : f_{mnk} \to^{r\lambda^\alpha} f\}$$

is called the Wijsman $r\lambda^\alpha$- statistical convergence limit set of the triple sequences of functions.

Definition

A triple sequence of functions (f_{mnk}) and $\alpha\in(0,1]$ said to be Wijsman $r\lambda^\alpha$- convergent to the function f denoted by $f_{mnk} \to^{r\lambda\alpha} f$, if $LIM^{r\lambda\alpha}f \neq \varphi$. In this case, $r\lambda^\alpha$ is called the Wijsman $r\lambda^\alpha$ convergent to the functions of degree of the triple sequence of functions $f=(f_{mnk})$. For r=0, we get the ordinary convergence.

Definition

A triple sequence of functions (f_{mnk}) and $\alpha\in(0,1]$ said to be Wijsman $r\lambda^\alpha$ - convergent to the functions f denoted by $f_{mnk} \to^{r\lambda\alpha} f$, provided that the set

$$\lim_{rst} 1/(\lambda^\alpha_{rst})|\{(m,n,k)\in I_{rst}: |d(f_{mnk},x)-d(f,x)|\geq r+\varepsilon\}|=0$$

has natural density zero for every $\varepsilon>0$, or equivalently, if the condition

$$\text{st-limsup}|d(f_{mnk},x)-d(f,x)|\leq r$$

is satisfied.

In addition, we can write $f_{mnk} \to^{r\lambda\alpha\ st} f$ if and only if the inequality

$$\lim_{rst} 1/(\lambda^\alpha_{rst})|\{(m,n,k)\in I_{rst}: |d(f_{mnk},x)-d(f,x)| < r+\varepsilon\}|=0$$

holds for every $\varepsilon>0$ and almost all (m,n,k). Here $r\lambda^\alpha$ is called the wijsman $r\lambda^\alpha$ roughness of degree. If we take r=0, then we obtain the ordinary Wijsman statistical convergence of triple sequence of functions.

In a similar fashion to the idea of classic Wijsman $r\lambda^\alpha$ rough convergence, the idea of Wijsman $r\lambda^\alpha$ rough statistical convergence to the triple sequence spaces of functions can be interpreted as follows:

Assume that a triple sequence of functions $=(f_{mnk})$ is Wijsman $r\lambda^\alpha$ statistically convergent to the functions and cannot be measured or calculated exactly; one has to do with an approximated (or Wijsman $r\lambda^\alpha$ statistically approximated) triple sequence of functions of $f=(f_{mnk})$ satisfying $|d(f_{mnk},x-y)-d(f,x-y)|\leq r$ for all m,n,k (or for almost all (m,n,k), i.e.,

$$\delta(\lim_{rst} 1/(\lambda^\alpha_{rst})|\{(m,n,k)\in I\_rst: |d(f_{mnk},x-y)-d(f,x-y)| > r\}|)=0.$$

Then the triple sequence of functions f_{mnk} is not $r\lambda^\alpha$ statistically convergent to the functions any more, but as the inclusion

$$\lim_{rst} 1/(\lambda^\alpha_{rst})\{|d(f\_mnk,y)-d(f,y)|\geq\varepsilon\}\supseteq\lim_{rst} 1/(\lambda^\alpha_{rst})\{|d(f_{mnk},x)-d(f,x)|\geq r+\varepsilon\} \quad ...(2.1)$$

holds and we have

$$\delta(\lim_{rst} 1/(\lambda^\alpha_{rst})|\{(m,n,k)\in I_{rst}: |d(f_{mnk},y)-d(f,y)|\geq\varepsilon\}|)=0,$$

i.e., we get

$$\delta(\lim_{rst} 1/(\lambda^\alpha_{rst})|\{(m,n,k)\in I_{rst}: |d(f_{mnk},x)-d(f,x)|\geq r+\varepsilon\}|)=0,$$

i.e., the triple sequence spaces of functions of f_{mnk} is Wijsman $r\lambda^\alpha$- statistically convergent to the functions in the sense of definition (2.3)

In general, the Wijsman rough $r\lambda^\alpha$ statistical convergence to the functions of limit of a triple sequence of functions may not unique for the Wijsman roughness degree r>0. So we have to consider the so called Wijsman $r\lambda^\alpha$- statistical convergence to the functions of limit set of a triple sequence of functions of (f_{mnk}), which is defined by

$$\text{st-LIM}(r\lambda^{\alpha}) f_{mnk}=\{f\in R: f_{mnk} \to^{r\lambda\alpha} \text{st}) f\}.$$

The triple sequence of functions of f_mnk is said to be Wijsman $r\lambda^{\alpha}$- statistically convergent to the functions provided that st-LIM($r\lambda\alpha$) $f_{mnk}\neq\varphi$. It is clear that if st-LIM($r\lambda^{\alpha}$) $f_{mnk}\neq\varphi$ for a triple sequence of functions (f_{mnk}) of real numbers, then we have

$$\text{st-LIM}^{\wedge}(r\lambda^{\alpha}) f_{mnk}=[\text{st-limsup} f_{mnk}-r, \text{st-liminf} f_{mnk}+r] \quad ...(2.2)$$

We know that $LIM^{r\lambda\alpha}=\varphi$ for an unbounded triple sequence of functions of (f_{mnk}). But such a triple sequence of functions of might be Wijsman $r\lambda^{\alpha}$ rough statistically convergent to the functions. For instance, define

$$d(f_{mnk},x)=\begin{cases}(-1)^{mnk}, & if (m,n,k)\neq(i,j,\ell)^2 (i,j,\ell\in\mathbb{N}),\\ (mnk), & otherwise\end{cases}$$

in R. Because the set {1,64,739,⋯} has natural density zero, we have

$$st-LIM^{r\lambda^{\alpha}} f_{mnk}=\begin{cases}\phi, & if r<1,\\ [1-r,r-1], & otherwise\end{cases}$$

and $LIM^{r\lambda\alpha} f_{mnk}=\varphi$ for all $r\geq 0$.

As can be seen by the example above, the fact that st-$LIM^{r\lambda\alpha}$) $f_{mnk}\neq\varphi$ does not imply $LIM^{r\lambda\alpha}$ $f_{mnk}\neq\varphi$. Because a finite set of natural numbers has natural density zero, $LIM^{r\lambda\alpha} f_{mnk}\neq\varphi$ implies st-$LIM^{\rho\lambda\alpha}$) $f_{mnk}\neq\varphi$.Therefore, we get $LIM^{\rho\lambda\alpha}$) $f_{mnk}\subseteq$st-$LIM^{\rho\lambda\perp\alpha}$ f_{mnk}. This obvious fact means $\{r\geq 0:LIM^{\lambda\perp\alpha}) f_{mnk}\neq\varphi\}\subseteq\{r\geq 0:\text{st-}LIM^{\rho\lambda\alpha}) f\_mnk\neq\varphi\}$ in this language of sets and yields immediately

$$\inf\{r\geq 0:LIM^{r\lambda\alpha} f_{mnk}\neq\varphi\}\geq\inf\{r\geq 0:\text{st-}LIM^{r\lambda\alpha} f_{mnk}\neq\varphi\}.$$

Moreover, it also yields directly diam($LIM^{\rho\lambda\alpha}$ f_{mnk})≤diam(st-$LIM^{r\lambda\alpha}$) f_{mnk}).

RESULTS

Theorem

A triple sequence of functions (f_{mnk}) and $\alpha\in(0,1]$ be any real number of Wijsman $r\lambda^{\alpha}$ statistically convergence to the functions, we have diam(st-$LIM^{\rho\lambda\alpha}$ f_{mnk})≤2r. In general diam(st-$LIM^{\rho\lambda\alpha}$ f_{mnk}) has an upper bound.

Proof

Assume that diam(st-$LIM^{\rho\lambda\alpha}$) f_{mnk})>2r. Then there exist w,y∈ st-$LIM^{\rho\lambda\alpha}$) f_{mnk} such that |w-y|>2r. Take $\varepsilon\in(0,|w-y|/2-r)$. Because w,y∈ st-$LIM^{\rho\lambda\alpha}$) f_{mnk}, we have $\delta(K_1)=0$ and $\delta(K_2)=0$ for every $\varepsilon>0$ where

$$K_1=\lim_{rst} 1/(\lambda^{\alpha}_{rst})|\{(m,n,k)\in I_{rst}:|d(f_{mnk},x)-d(f,w)|\geq r+\varepsilon\}|=0 \text{ and}$$

$$K_2=\lim_{rst} 1/(\lambda^{\alpha}_{rst})|\{(m,n,k)\in I_{rst}:|d(f_{mnk},x)-d(f,y)|\geq r+\varepsilon\}|=0.$$

Using the properties of natural density, we get $\delta(K^c_1\cap K^c_2)=1$. Thus we can write

$$|w-y|\leq|d(f_{mnk},x)-d(f,w)|+|d(f_{mnk},x)-d(f,y)|$$
$$<2(r+\varepsilon)=2(|w-y|/2)=|w-y|$$

for all $(m,n,k)\in K^c_1\cap K^c_2$, which is a contradiction.

Now let us prove the second part of the theorem. Consider a Wijsman triple sequence of functions of real numbers of (f_{mnk}) such that st-$\lim f_{mnk}=f$. Let $\varepsilon>0$. Then we can write

$$\delta(\lim_{rst} 1/(\lambda^{\alpha}_{rst})|\{(m,n,k)\in I_{rst}:|d(f_{mnk},x)-d(f,x)|\geq\varepsilon\}|)=0.$$

We have

$$|d(f_{mnk},x)-d(f,y)|\leq|d(f_{mnk},x)-d(f,x)|+|d(f,y)-d(f,y)|\leq|d(f_{mnk},x)-d(f,x)|+r$$

for each $$d(f_{mnk},y)\in\bar{B}_{r\lambda^{\alpha}}(f)=lim_{rst}\frac{1}{\lambda^{\alpha}_{rst}}|\{(m,n,k)\in I_{rst}:|d(f_{mnk},y)-d(f,x)|\leq r\}|$$

Then we get $|d(f,y)-d(f_{mnk},y)|<r+\varepsilon$ for each

$(m,n,k)\in\lim_{rst} 1/(\lambda^{\alpha}_{rst})|\{(m,n,k)\in I_{rst}:|d(f_{mnk},x)-d(f,x)|<\varepsilon\}|=0$. Because the triple sequence of functions of real numbers of f_{mnk} is Wijsman statistically convergent to the functions of f, we have

$$\delta(\lim_{rst} 1/(\lambda^{\alpha}_{rst})|\{(m,n,k)\in I_{rst}:|d(f_{mnk},x)-d(f,x)|<\varepsilon\}|)=1$$

Therefore we get y∈ st-$LIM^{r\lambda\alpha}$ f_{mnk}. Hence, we can write

$$st-LIM^{r\lambda^{\alpha}} f_{mnk}=\bar{B}_{r\lambda^{\alpha}}(f).$$

Because $diam(\bar{B}_{r\lambda^{\alpha}}(f)) = 2r$, this shows that in general, the upper bound 2r of the diameter of the set st-LIM$^{r\lambda\alpha}$) f_{mnk} is not an lower bound.

Theorem

Let r>0. A triple sequence of functions (f_{mnk} and α∈(0,1] be any real number of (d(f_{mnk},x) is Wijsman rλ^{α}- statistically convergent to the functions of f if and only if there exists a triple sequence of functions (d(f_{mnk},y) such that st-limf_{mnk}=f and |d(f_{mnk},x-y)-d(f,x-y) |≤r for each (m,n,k)∈ I_{rst}.

Proof

Necessity: Assume that $f_{mnk} \rightarrow^{rst}$ f.
Then we have
st-limsup|d(f_{mnk},x)-d(f,x) |≤r. ...(3.1)

Now, define

$$d(f_{mnk},y) = \begin{cases} f, & if|d(f_{mnk},x) - d(f,x)| \le r, \\ d(f_{mnk},x) + r\left(\frac{d(f,x) - d(f_{mnk},x)}{|d(f_{mnk},x) - d(f,x)|}\right), & otherwise, \end{cases}$$

$$|d(f_{mnk},y) - d(f,y)| = \begin{cases} |f - f|, & if|d(f_{mnk},x) - d(f,x)| \le r, \\ |d(f_{mnk},x) - d(f,x)| + r\left(\frac{|f - f| - |d(f_{mnk},x) - d(f,x)|}{|d(f_{mnk},x) - d(f,x)|}\right), & otherwise, \end{cases}$$

$$|d(f_{mnk},y) - d(f,y)| = \begin{cases} 0, & if|d(f_{mnk},x) - d(f,x)| \le r, \\ |d(f_{mnk},x) - d(f,x)| - r\left(\frac{|d(f_{mnk},x) - d(f,x)|}{|d(f_{mnk},x) - d(f,x)|}\right), & otherwise, \end{cases}$$

$$|d(f_{mnk},y) - d(f,y)| = \begin{cases} 0, & if|d(f_{mnk},x) - d(f,x)| \le r, \\ |d(f_{mnk},x) - d(f,x)| - r, & otherwise. \end{cases}$$

We have |d(f_mnk,y)-d(f,y) |≥|d(f_mnk,x)-d(f,x) |-rðd(f_mnk,x)-d(f,x)-d(f_mnk,y)+d(f,y) |≤r

|d(f_mnk,x-y) |≤r ...(3.2)

for all (m,n,k)ð_rst. By equation (3.1) and by definition of d(f_{mnk},y), we get st-limsup|d(f_{mnk},y)-d(f,y) |=0.

⇒st-lim$f_{mnk} \rightarrow^{r\lambda\alpha}$ f.

Sufficiency

Because st-limf_{mnk}=f, we have

δ($\lim_{rst}$ 1/(λ^{α}_{rst}) |{(m,n,k)∈ I_{rst}:|d(f_{mnk},y)-d(f,y) |≥ε}|)=0

for each ε>0. It is easy to see that the inclusion

{(m,n,k)∈ I_{rst}:|d(f_mnk,y)-d(f,y) |≥ε}
⊇{(m,n,k)∈ I_{rst}:|d(f_{mnk},x)-d(f,x) |≥r+ε}

holds. Because δ($\lim_{rst}$ 1/(λ_{rst}^{α}) |{(m,n,k)∈ I_{rst}:|d(f_{mnk},y)-d(f,y) |≥ε} |)=0, we get

δ($\lim_{rst}$1/(λ^{α}_{rst})|{(m,n,k)∈ I_{rst}:|d(f_{mnk},x)-d(f,x) |≥r+ε}|)=0.

Remark

If we replace the condition |d(f_{mnk},x-y) |≤r for all (m,n,k)∈ I_{rst} in the hypothesis of the Theorem (3.2) with the condition

δ($\lim_{rst}$ 1/(λ^{α}_{rst}) |{(m,n,k)∈ I_{rst}:|d(f_{mnk},x-y) |>r} |)=0
is valid.

Theorem

For an arbitrary c∈ Γ_x of Wijsman rλ^{α} statistically convergence to the functions of (d(f_{mnk},x)) we have |f-c|≤r for all f∈ st-LIM$^{r\lambda\alpha}$) f_{mnk}.

Proof

Assume on the contrary that there exist a point c∈ Γ_x and f∈ st-LIM$^{r\lambda\alpha}$ f_{mnk} such that |f-c|>r. Define ε:=(|f-c|-r)/3. Then
{(m,n,k)∈ I_{rst}:|d(f,x)-c|<ε}⊆{(m,n,k)∈ I_{rst}:|d(f_{mnk},x)-d(f,x) |≥r+ε} ...(3.3)

Since c∈ Γ_x, we have

δ($\lim_{rst}$ 1/(λ^{α}_{rst}) {(m,n,k)∈ I_{rst}:|d(f_{mnk},x)-c|<ε})≠0.

Hence, by (3.3), we get

δ($\lim_{rst}$ 1/(λ^{α}_{rst}) |{(m,n,k)∈ I_{rst}:|d(f_{mnk},x)-d(f,x) |≥r+ε} |)≠0,

which contradicts the fact $f \in st - LIM^{r\lambda^{\alpha}}$$f_{mnk}$.

Proposition

If a Wijsman rλ^{α} statistically convergence to the functions of (d(f_{mnk},x)) is analytic, then there exists a non-negative real number r such that $st - LIM^{r\lambda^{\alpha}} f_{mnk} \neq \phi$.

Proof

If we take the Wijsman rλ^{α} statistically convergence to the functions to be Wijsman rλ^{α} statistically convergence to the functions of analytic, then the of proposition holds. Thus we have the following theorem.

Theorem

A triple sequence of functions (f_{mnk}) and $\alpha \in (0,1]$ be any real number of ($d(f_{mnk},x)$) is Wijsman $r\lambda^{\alpha}$ statistically convergence to the functions of analytic if and only if there exists a non-negative real number r such that $st - LIM^{r\lambda^{\alpha}} f_{mnk} \neq \phi$.

Proof

Since the triple sequence of functions of $d(f_{mnk},x)$ is Wijsman $r\lambda^{\alpha}$ statistically convergence to the functions of analytic, there exists a positive real number M such that

$$\delta\left(lim_{rst} \frac{1}{\lambda_{rst}^{\alpha}} \left|\{(m,n,k) \in I_{rst} : |d(f_{mnk},x)|^{1/m+n+k} \geq M\}\right|\right) = 0$$

Define

$$r' = sup\{|d(f_{mnk},x)|^{1/m+n+k} : (m,n,k) \in I_{rst}\}.$$

$$K = lim_{rst} \frac{1}{\lambda_{rst}^{\alpha}} \left|\{(m,n,k) \in I_{rst} : |d(f_{mnk},x)|^{1/m+n+k} \geq M\}\right|$$

Then the set $st - LIM^{r\lambda^{\alpha'}} f_{mnk}$ contains the origin of R. So we have $st - LIM^{r\lambda^{\alpha}} f_{mnk} \neq \phi$.

If $st - LIM^{r\lambda^{\alpha}} f_{mnk} \neq \phi$ for some $r \geq 0$, then there exists d(f,x) such that $f \in st - LIM^{r\lambda^{\alpha}} f_{mnk}$, f_mnk, i.e.,

$$\delta\left(lim_{rst} \frac{1}{\lambda_{rst}^{\alpha}} \left|\{(m,n,k) \in I_{rst} : |d(f_{mnk},x) - d(f,x)|^{1/m+n+k} \geq r + \varepsilon\}\right|\right) = 0$$

for each $\varepsilon > 0$. Then we say that almost all $d(f_{mnk},x)$ are contained in some ball with any radius greater than r. So the triple sequence of functions of $d(f_{mnk},x)$ is Wijsman $r\lambda^{\alpha}$ statistically convergence to the functions of analytic.

Remark

If $f' = (f_{m_i n_j k_\ell})$ is a sub sequence of functions of (f_{mnk}), then $LIM^{r\lambda^{\alpha}} f_{mnk} \subseteq LIM^{r\lambda^{\alpha}} f'_{mnk}$. But it is not valid for Wijsman $r\lambda^{\alpha}$ statistical convergence to the functions. For Example: Define

$$d(f_{mnk},x) = \begin{cases} (mnk), & if (m,n,k) = (i,j,\ell)^2 (i,j,\ell \in I_{rst}), \\ 0, & otherwise \end{cases}$$

of real numbers. Then the triple sequence of functions of f'=(1,64,739,...) is a sub sequence of functions of f. We have $st - LIM^{r\lambda^{\alpha}} f_{mnk} = [-r,r]$ and $st - LIM^{r\lambda^{\alpha}} f'_{mnk} = \phi$.

Theorem

Let $f' = (f_{m_i n_j k_\ell})$ is a non thin sub sequence of functions of Wijsman $r\lambda^{\alpha}$ statistically convergence to the functions of $f=(f_{mnk})$, then $st - LIM^{r\lambda^{\alpha}} f_{mnk} \subseteq st - LIM^{r\lambda^{\alpha}} f'_{mnk}$

Proof: Omitted.

Theorem

The Wijsman $r\lambda^{\alpha}$- statistical convergence to the functions of limit set of a triple sequence to the functions of (f_{mnk}) is closed.

Proof

If $st - LIM^{r\lambda^{\alpha}} f_{mnk} \neq \phi$, then it is true. Assume that $st - LIM^{r\lambda^{\alpha}} f_{mnk} \neq \phi$, then we can choose a triple sequence of functions of $(d(f_{mnk},y)) \subseteq st - LIM^{r\lambda^{\alpha}} f_{mnk}$ such that $d(f_{mnk},y) \to^{r\lambda^{\alpha}} d(f,y)$ as $m,n,k \to \infty$. If we prove that $f \in st - LIM^{r\lambda^{\alpha}} f_{mnk}$, then the proof will be complete.

Let $\varepsilon > 0$ be given. Because $d(f_{mnk},y) \to^{r\lambda^{\alpha}} d(f,y), \exists \left(m_{\frac{\varepsilon}{2}}, n_{\frac{\varepsilon}{2}}, k_{\frac{\varepsilon}{2}}\right) \in I_{rst}$

such that

$|d(f_{mnk},y) - d(f,y)| < \frac{\varepsilon}{2}$ for all $m > m_{\frac{\varepsilon}{2}}, n > n_{\frac{\varepsilon}{2}}, k > k_{\frac{\varepsilon}{2}}$.

Now choose an $(m_0\ n_0\ k_0) \in I_{rst}$ such that $m_0 > m\varepsilon/2, n_0 > n\varepsilon/2, k_0 > k\varepsilon/2$. Then we can write

$$\left|d(f_{m_0 n_0 k_0},y) - d(f,y)\right| < \frac{\varepsilon}{2}.$$

On the other hand, because $(d(f_{mnk},y)) \subseteq st - LIM^{r\lambda^{\alpha}} f_{mnk}$, we have $d(f_{m_0 n_0 k_0},y) \in st - LIM^{r\lambda^{\alpha}} f_{mnk}$ namely,

$$\delta\left(lim_{rst} \frac{1}{\lambda_{rst}^{\alpha}} \left|\left\{(m,n,k) \in I_{rst} : \left|d(f_{mnk},x) - d(f_{m_0 n_0 k_0},y)\right| \geq r + \frac{\varepsilon}{2}\right\}\right|\right) = 0 \quad ...(3.4)$$

Now let us show that the inclusion

$$\{|d(f_{mnk},x) - d(f,x)| < r + \varepsilon\} \supseteq \left\{\left|d(f_{mnk},x) - d(f_{m_0 n_0 k_0},y)\right| < r + \frac{\varepsilon}{2}\right\} \quad ...(3.5)$$

holds. Take

$(i,j,\ell) \in \left\{(m,n,k \in I_{rst}) : \left|d(f_{mnk},x) - d(f_{m_0 n_0 k_0},y)\right| < r + \frac{\varepsilon}{2}\right\}$

Then we have

$\left|d(f_{mnk},x) - d(f_{m_0 n_0 k_0},y)\right| < r + \frac{\varepsilon}{2}$

and hence

$$\left|d(f_{ij\ell},x) - d(f,y)\right| \leq \left|d(f_{ij\ell},x) - d(f_{m_0 n_0 k_0},y)\right| +$$

$$\left|d(f_{m_0 n_0 k_0},y) - d(f_0,y)\right| < r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < r + \varepsilon$$

i.e., (i,j,l)∈{(m,n,k)∈I_{rst}:|d(f_{mnk},x)-d(f,x) |<r+ε} which proves the equation (3.5). Hence the natural density of the set on the LHS of equation (3.5) is equal to 1. So we get δ(lim_{rst} 1/(λ^{α}_{rst}) |{(m,n,k)∈I_{rst}:|d(f_{mnk},x)-d(f,x) |≥r+ε} |)=0.

Theorem

The Wijsman rλ^α- statistical convergence to the functions of limit set of a triple sequence of functions of (d(f_{mnk},x)) is convex.

Proof

Let $f_1, f_2 \in st - LIM^{r\lambda^{\alpha}} f_{mnk}$ for the triple sequence of functions of (d(f_{mnk},x)) and let ε>0 be given. Define K_1=lim_{rst} 1/(λ^{α}_{rst}) |{(m,n,k)∈I_{rst}:|d(f_{mnk},x)-d(f_1,x) |≥r+ε} |=0 and

$$K_2 = lim_{rst} \frac{1}{\lambda^{\alpha}_{rst}} |\{(m,n,k) \in I_{rst}: |d(f_{mnk}, x) - d(f_2, x)| \geq r + \varepsilon\}| = 0.$$

Because $f_1, f_2 \in st - LIM^{r\lambda^{\alpha}} f_{mnk}$, we have $\delta(K_1) = \delta(K_2) = 0$. Thus we have

$$|d(f_{mnk}, x) - [(1-\lambda^{\alpha})d(f_1, x) + \lambda d(f_2, x)]|$$
$$= |(1-\lambda^{\alpha})(d(f_{mnk}, x) - d(f_1, x)) + \lambda(d(f_{mnk}, x) - d(f_2, x))| < r + \varepsilon,$$

for each $(m, n, k) \in (K_1^c \cap K_2^c)$ and each λ∈[0,1], and α∈(0,1]. Because $\delta(K_1^c \cap K_2^c) = 1$,, we get

$$\delta\left(lim_{rst} \frac{1}{\lambda^{\alpha}_{rst}} \left|\left\{(m,n,k) \in I_{rst}: \left|d(f_{mnk}, x) - \left[\begin{matrix}(1-\lambda^{\alpha})d(f_1, x)\\ +\lambda^{\alpha} d(f_2, x)\end{matrix}\right]\right| \geq r + \varepsilon\right\}\right|\right) = 0$$

i.e., $[(1-\lambda^{\alpha})d(f_1, x) + \lambda d(f_2, x)] \in st - LIM^{r\lambda^{\alpha}} f_{mnk}$, which proves the convexity of the set $st - LIM^{r\lambda^{\alpha}} f_{mnk}$

Theorem

A triple sequence of functions (f_{mnk}) and α∈(0,1] be any real number of (d(f_{mnk},x)) of Wijsman rλ^α statistically converges to the functions of f if and only if $st - LIM^{r\lambda^{\alpha}}(d(f_{mnk}, x)) = \bar{B}_{r\lambda^{\alpha}}(d(f, x))$.

Proof

We have proved the necessity part of this theorem in proof of the Theorem (3.1).

Sufficiency

Because $st - LIM^{r\lambda^{\alpha}}(d(f_{mnk}, x)) = \bar{B}_{r\lambda^{\alpha}}(d(f, x)) \neq \phi$, then by Theorem (3.5) we can say that the triple sequence spaces d(f_{mnk},x) is Wijsman rλ^α statistically convergence to the functions of analytic. Assume on the contrary that the triple sequence of functions of (d(f,x)) has another Wijsman rλ^α statistical convergence to the functions of cluster point (d(f,x))' different from d(f,x). Then the point

$$(d(f,x)) = d(f,x) + \frac{r\lambda}{|d(f,x) - (d(f,x))'|}(d(f,x) - (d(f,x))')$$

satisfies

$$(d(f,x)) - (d(f,x))'$$
$$= d(f,x) - (d(f,x))' + \frac{r\lambda^{\alpha}}{|d(f,x) - (d(f,x))'|}(d(f,x) - (d(f,x))')$$
$$|(d(f,x)) - (d(f,x))'|$$
$$= |d(f,x) - (d(f,x))'| + \frac{r\lambda^{\alpha}}{|d(f,x) - (d(f,x))'|}(d(f,x) - (d(f,x))')$$
$$|d(f,x) - (d(f,x))'| = |d(f,x) - (d(f,x))'| + r > r.$$

Because (d(f,x))') is a Wijsman rλ^α statistical convergence to the functions of cluster point of the triple sequence of functions of d(f_{mnk},x), by Theorem (2.4) this inequality implies that $d(f,x) \notin st - LIM^{r\lambda^{\alpha}}(d(f,x))$. This contradicts the fact $|d(f,x) - d(f,x)| = r$ and $st - LIM^{r\lambda^{\alpha}}(d(f,x)) = \bar{B}_{r\lambda^{\alpha}}(d(f,x))$.Therefore, d(f,x) is the unique Wijsman rλ^α statistical convergence to the functions of cluster point of the triple sequence of functions of d(f_{mnk},x). Hence the Wijsman rλ^α statistical convergence to the functions of cluster point of a Wijsman rλ^α statistically convergence to the functions of analytic is unique, then the triple sequence of functions d(f_{mnk},x) is wijsman rλ^α statistically convergent to the functions of d(f,x).

Theorem

Let (R^3,|.,.|) be a strictly convex space and (d(f_{mnk},x)) be a triple sequence of functions and α∈(0,1] be any real number, if there exist $y_1, y_2 \in st - LIM^{r\lambda^{\alpha}}(d(f,x))$such that $|d(f,y_1) - d(f,y_2)| = 2r$ then this triple sequence of functions is Wijsman rλ^α statistically convergent to the functions of 1/2 (d(f,y_1)+d(f,y_2)).

Proof

Assume that $z \in \Gamma_{(d(f_{mnk},x))}$. Then y_1,y_2∈ $st - LIM^{r\lambda^{\alpha}}(d(f_{mnk}, x))$ (d(f_{mnk},x)) implies that

|(d(f,y_1))-z|≤ r and|(d(f,y_2))-z|≤r ...(3.6)

by Theorem 3.4. On the other hand, we have

2r=|(d(f,y_1))-(d(f,y_2)) |≤|(d(f,y_1))-z|+|(d(f,y_2))-z| ...(3.7) combining the inqualities (3.6) and (3.7), we get $|(d(f,y_1))-z|=|(d(f,y_2))-z|=r$. Because

$$1/2\ ((d(f,y_2))-(d(f,y_1)))=1/2\ [(z-(d(f,y_1)))+(-z+(d(f,y_2)))] \quad ...(3.8)$$

and $|(d(f,y_1))-(d(f,y_2))|=2r$, we get $|1/2\ ((d(f,y_2))-(d(f,y_1)))|=r$. By the strict convexity of the space and from the equality 3.8, we get

$1/2\ ((d(f,y_2))-(d(f,y_1)))=(z-(d(f,y_1)))=(-z+(d(f,y_2)))$ which implies that $z=1/2((d(f,y_1))+(d(f,y_2)))$. Hence z is the unique Wijsman $r\lambda^\alpha$ statistical convergence to the functions of cluster point of the triple sequence of functions of (d(f,x)).

On the other hand, the assumption $y_1,y_2 \in st-LIM^{r\lambda^\alpha}(d(f_{mnk},y))$ implies that $st-LIM^{r\lambda^\alpha} f \neq 0$. By Theorem 3.6, the triple sequence of functions of $(d(f_{mnk},y))$ is $r\lambda^\alpha$ statistically convergence to the functions of analytic. Consequently, the statistical convergence to the functions of cluster point of a Wijsman $r\lambda^\alpha$ statistically convergence to the functions of analytic is unique, then the triple sequence of functions is Wijsman $r\lambda^\alpha$ statistically convergent to the functions of $st\text{-}\lim(d(f_{mnk},y))=1/2\ ((d(f,y_1))+(d(f,y_2)))$.

Theorem

(a) If $c\in\Gamma_x$ then

$$st-LIM^{r\lambda^\alpha}(d(f_{mnk},x)) \subseteq \bar{B}_{r\lambda^\alpha}(c) \quad ...(3.9)$$

(b)
$$st-LIM^{r\lambda^\alpha}(d(f_{mnk},x)) = \cap_{c\in\Gamma_x} \bar{B}_{r\lambda^\alpha}(c) = \left\{(d(f,x)) \in \mathbb{R}^3 : \Gamma_x \subseteq \bar{B}_{r\lambda^\alpha}\left((d(f,x))\right)\right\} \quad ...(3.10)$$

Proof

(a) Assume that $(d(f,x)) \in st-LIM^{r\lambda^\alpha}(d(f_{mnk},x))$ and $c\in\Gamma_x$. Then by Theorem 3.4, we have

$$|(d(f,x))-c|\le r;$$

other wise we get

$$\delta\left(\lim_{rst}\frac{1}{\lambda^\alpha_{rst}}\{(m,n,k)\in I_{rst}: |(d(f_{mnk},x))-(d(f,x))| \ge r+\varepsilon\}\right) \neq 0$$

for $\varepsilon=(|(d(f,x))-c|-r)/3$. This contradicts the fact $(d(f,x))\in LIM^{r\lambda^\alpha}(d(f_{mnk},x))$.

(b) By the equation (3.9), we can write

$$st-LIM^{r\lambda^\alpha}(d(f_{mnk},x)) \subseteq \cap_{c\in\Gamma_x} \bar{B}_{r\lambda^\alpha}(c) \quad ...(3.11)$$

Now assume that $A \in \cap_{c\in\Gamma_x} \bar{B}_{r\lambda^\alpha}(c)$. Then we have

$$|(d(f,y))-c|\le r$$

for all $c\in\Gamma_x$, which is equivalent to $\Gamma_x \subseteq \bar{B}_{r\lambda^\alpha}\left((d(f,y))\right)$ i.e.,

$$\cap_{c\in\Gamma_x} \bar{B}_{r\lambda^\alpha}(c) \subseteq \left\{(d(f,y)) \in \mathbb{R}^3 : \Gamma_x \subseteq \bar{B}_{r\lambda^\alpha}\left((d(f,y))\right)\right\}. \quad ...(3.12)$$

Now let $(d(f,y)) \notin st\text{-}LIM^{r\lambda^\alpha}(d(f_{mnk},x))$. Then there exists an $\varepsilon>0$ such that

$$\delta\left(\lim_{rst}\frac{1}{\lambda^\alpha_{rst}}\{(m,n,k)\in I_{rst}: |(d(f,x))-(d(f,y))| \ge r+\varepsilon\}\right) \neq 0$$

the existence of a Wijsman $r\lambda^\alpha$ statistical convergence to the functions of cluster point c of the triple sequence of functions of $(d(f_{mnk},x))$ with $|(d(f,y))-c|\ge r+\varepsilon$, i.e., Γ_x Ú$\bar{B}_{r\lambda^\alpha}$ ((d(f,y))) and

$$(d(f_{mnk},y)) \notin \left\{f \in \mathbb{R}^3 : \Gamma_x \subseteq \bar{B}_{r\lambda^\alpha}\left((d(f,x))\right)\right\}.$$

Hence $d(f_{mnk},y) \in st-LIM^{r\lambda^\alpha}(d(f_{mnk},x))$ follows from

$d(f_{mnk},y) \in \left\{(d(f,x)) \in \mathbb{R}^3 : \Gamma_x \subseteq \bar{B}_{r\lambda^\alpha}\left((d(f,x))\right)\right\}$, i.e.,

$$\left\{(d(f,x)) \in \mathbb{R}^3 : \Gamma_x \subseteq \bar{B}_{r\lambda^\alpha}\left((d(f,x))\right)\right\} \subseteq st-LIM^{r\lambda^\alpha}(d(f_{mnk},x)) \quad ...(3.13)$$

Therefore the inclusions (3.11)-(3.13) ensure that (3.10) holds.

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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