ISSN: 2456-799X, Vol.02, No.(2) 2017, Pg. 64-70

Oriental Journal of Physical Sciences

www.orientjphysicalsciences.org

Fuzzy Logic Models for Evaluating Student Understanding of Polar Coordinates

MICHAEL GR. VOSKOGLOU

Graduate Technological Educational Institute of Western Greece School of Technological Applications, 26334 Patras, Greece.

Abstract

In the paper at hands two assessment methods based on principles of *Fuzzy* Logic are applied for evaluating the student understanding of polar coordinates on the plane. The first of them utilizes *triangular fuzzy* numbers as assessment tools and focuses on student mean performance, while the second one adapts properly the Centre of Gravity defuzzification technique to measure the student quality performance. The connections and differences of these methods with the traditional assessment methods of calculating the mean values of the student scores and the Grade Point Average index respectively are also discussed and a classroom experiment performed in an earlier work is reused to illustrate the applicability of the above methods for the purposes of the present work.

Article History

Received: 04 August 2017
Accepted: 06 October 2017

Keywords:

Polar Coordinates on the Plane, Fuzzy Logic (FL), Triangular Fuzzy Numbers (TFNs), Centre of Gravity (COG) Defuzzification Technique, Grade Point Average (GPA) Index, Student Assessment.

Introduction

This work evaluates the student difficulties for the understanding and proper use of polar coordinates in the plane (see Figure 1) using assessment methods of the *Fuzzy* Logic (FL). The rest of the article is organized as follows: In Section 2 a synopsis is presented of the traditional and *fuzzy logic* methods that we have used in earlier works for assessing a system's performance. In Section 3 *Triangular Fuzzy Numbers* (TFNs) are used as assessment tools of a student group mean performance, while in Section 4 the *Centre of Gravity* (*COG*) defuzzification technique is properly adapted for measuring a student group quality performance. As an application, in Section 5 the data of a classroom experiment performed in an earlier work are reused for the purposes of the present study. In Section 6 the outcomes of the assessment methods that have been utilized in the previous Section are compared to each other and the last Section 7 is devoted to the final conclusion and to a proposal for future research on the subject.

CONTACT Michael Gr. Voskoglou mvosk@hol.gr Graduate Technological Educational Institute of Western Greece School of Technological Applications, 26334 Patras, Greece.

Traditional and Fuzzy Assessment Methods

The traditional method for assessing a student group mean performance is the calculation of the mean value of the student numerical scores (marks). However, frequently in practice the student individual performance is evaluated not by numerical scores, but by qualitative characterizations (grades), like excellent, very good, good, fair, unsatisfactory, etc. In such cases we have considered in earlier works the measurement of a fuzzy system's *uncertainty* as a tool for assessing a student group mean performance (e.g. [1: Chapter 5, 2: Section 3, 3: Chapter 5], etc.). Nevertheless, this method, apart of requiring laborious calculations, can be used to compare the mean performance of two different student groups only under the assumption that the initially existing uncertainty for the two groups is the same, a condition which is not always true.

More recently we have also used *fuzzy numbers* as tools for evaluating a student group mean performance (e.g. [1: Chapter 7, 3: Chapter 8, 4, 5] etc.). This method, a special case of which will be applied in Section 3 of the present paper, appears to be more general and accurate than the measurement of the uncertainty.

Note also that, in certain cases, depending on the required goals, a group's assessment is focused on its *quality performance*, by assigning greater coefficients to the higher scores. A traditional assessment method of this kind is the *Grade Point Average (GPA) index*[6], which is calculated by the formula

$$\text{GPA} = \frac{0n_F + 1n_D + 2n_C + 3n_B + 4n_A}{n} \qquad \text{...(1)}$$

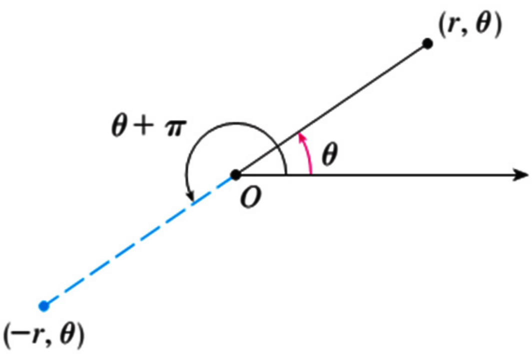

Fig.1: Polar coordinates of a point of the plane.

where n is the total number of the group's members and n_A, n_B, n_C, n_D and n_F denote the numbers of the group's members that demonstrated excellent (A), very good (B), good (C), fair (D) and unsatisfactory (F) performance respectively. Obviously $0 \leq$ GPA ≤ 4, therefore values of GPA greater than 2 are demonstrating a more than satisfactory performance. Note that formula (1) can be also written in the form

$$\text{GPA} = y_2 + 2y_3 + 3y_4 + 4y_5 \qquad \text{...(2),}$$

where $y_1 = \frac{n_F}{n}$, $y_2 = \frac{n_D}{n}$, $y_3 = \frac{n_C}{n}$, $y_4 = \frac{n_B}{n}$ and $y_5 = \frac{n_A}{n}$.

In earlier works (e.g. [1: Chapter 6, 2: Section 3, 3: Chapter 6, 7], etc.) an analogous method of FL was developed for measuring a student group quality performance by properly adapting the COG *defuzzification* technique. For the needs of the present paper this method will be sketched in Section 4.

Assessment of a Student Group Performance Using Triangular Fuzzy Numbers

For general facts on *Fuzzy Sets (FSs)* we refer to[8]. A *Fuzzy Number (FN)* is a special form of FS on the set R of real numbers. For general facts on FNs we refer to[9].

Definition: Triangular Fuzzy Number (TFN)

A TFN of the form A (a, b, c), with a, b, c real numbers such that a<c<b, is a FN with membership function defined by

$$y = m(x) = \begin{cases} \frac{x-a}{b-a}, & x \in [a,b] \\ \frac{c-x}{c-b}, & x \in [b,c] \\ 0, & x < a \text{ or } x > c \end{cases}$$

Let A (a, b, c) and B (a_1, b_1, c_1) be two TFNs and let k be a positive real number. Then, one can define:

- The sum A + B = (a+a_1, b+b_1, c+c_1)
- The scalar product kA = (ka, kb, kc)[9].

Further, for the needs of the present work, we give the following definition:

Definition: Mean value of TFNs
Given the TFNs A_i, i = 1, 2,…, n , where n is a non negative integer, n ≥ 2, their mean value is defined to be the TFN M = $(A_1 + A_2 + ... + A_n)$.

The assessment process of a student group mean performance using TFNs involves the following steps:

1. Numerical evaluation of each student's individual performance in a climax from 0 to 100.
2. Qualitative characterization of this performance by introducing the fuzzy linguistic labels (grades): A (85-100) = excellent, B (75-84) = very good, C (60-74) = good, D(50-59) = fair and F(0-49) = non satisfactory. Note that the above correspondence between the student scores and the linguistic grades, although it is compatible to the common logic, should not be considered as being unique. For example, in a more strict assessment one could take A (90-100), B (80-89), C (70-79), D (60-69), F(0-59), etc. Further, more linguistic grades could be added, like E (marginal success) between D and F, or B^+ between A and B, B^- between B and C, etc.
3. Assignment to each of the above grades of a TFN denoted, for reasons of simplicity, by the same letter as follows: A = (85, 92.5, 100), B = (75, 79.5, 84), C = (60, 67, 74), D = (50, 54.5, 59) and F = (0, 24.5, 49). Observe that the middle entry of each of those TFNs is equal to the mean value of the student scores attached to the corresponding grade. In this way a TFN corresponds to each student assessing his/her individual performance.
4. Calculation of the mean value *M* of all TFNs corresponding to each student's individual performance, to be used – as it is logical to do - as a fuzzy measure for evaluating the student group mean performance.
5. Defuzzification of M, in order to obtain a crisp representative of it, through which the conclusion about the student group evaluation can be drawn.

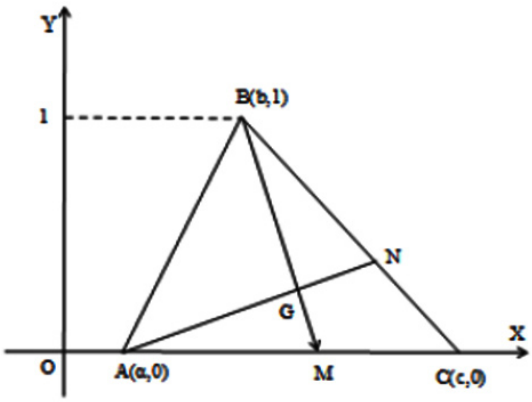

Fig. 2: Graph of the TFN T (a, b, c)

For the last step we need the following Lemma:
Lemma: Centre of Gravity (COG) of a TFN
Let T = (a, b, c) be a TFN. Then the coordinates of the COG of its graph are calculated by the formulas x(T) = $\frac{a+b+c}{3}$, y(T) = $\frac{1}{3}$.

Proof: From Definition 3.2 it becomes evident that the graph of T is the triangle ABC of Figure 2, with A(a, 0). B(b, 1) and C (c, 0).

The COG of the triangle ABC is the point O of the intersection of its medians AN and BM, Therefore, it is a routine to apply basics of Analytic Geometry in order to find the equations of the straight lines AN and BM and the coordinates of their intersection point G. As a consequence of the above Lemma one obtains the following result:

Theorem : Defuzzification of the mean value M If M (a, b, c) is the mean value of all TFNs corresponding to each student's individual performance, then the x-coordinate of the COG of M is given by x(M)= b.

Proof: If (a_i, b_i, c_i) are the TFNs A, B, C, D, F, with i = 1, 2, 3, 4, 5 respectively, then $b_i = .\frac{a_i+c_i}{2}$. Therefore,

$$\frac{a_i+b_i+c_i}{3}=\frac{3(a_i+c_i)}{6}=b_i \qquad ...(3)$$

According to Definition 3.2 the mean value M has the form M = $k_1A + k_2B + k_3C + k_4D + k_5F$, with k_i non negative rational numbers, i = 1, 2, 3, 4, 5. Consequently

$$M(a,\ b,\ c)=\sum_{i=1}^{5}k_i(a_i,b_i,c_i)=(\sum_{i=1}^{5}k_ia_i,\sum_{i=1}^{5}k_ib_i,\sum_{i=1}^{5}k_ic_i) \qquad ...(4)$$

Therefore, Lemma 3.4 gives that x(M) =

$$\frac{\sum_{i=1}^{5}k_ia_i+\sum_{i=1}^{5}k_ib_i+\sum_{i=1}^{5}k_ic_i}{3}=\sum_{i=1}^{5}k_i\frac{a_i+b_i+c_i}{3}.$$

Thus, applying equality (3), one finds that x(M)= $\sum_{i=1}^{5}k_ib_i$ and equality (4) gives that x(M) = b.

Assessment of a Student Group Performance Using the COG Defuzzification Technique (RFAM)

There is a commonly used in FL approach (e.g. see[10]) to represent a system's fuzzy data by the coordinates (x_c, y_c) of the COG, say F_c, of the level's area F contained between the graph of the membership function *y=m(x)* the OX axis, which can be calculated by using well-known[11] from Mechanics formulas.

Consider now the special case where one deals with the assessment of a group's performance Then, we choose as set of the discourse the set *U* = {A, B, C, D, F} of the *fuzzy linguistic labels* (characterizations) of excellent (A), very good (B), good (C), fair (D) and unsatisfactory (F) performance respectively of the group's members. When a score, say y, is assigned to a group's member (e.g. a mark in case of a student), then its performance is characterized by F, if y $\in$ [0, 1) , by D, if y $\in$ [1, 2), by C, if y $\in$ [2, 3), by B if y $\in$ [3, 4) and by A if y $\in$ [4, 5] respectively. Consequently, we have that y_1 = m(x) = m(F) for all x in [0,1), y_2 = m(x) = m(D) for all x in [1,2), y_3 = m(x) = m(C) for all x in [2, 3), y_4 = m(x) = m(B) for all x in [3, 4) and y_5 = m(x) = m(A) for all x in [4,5].

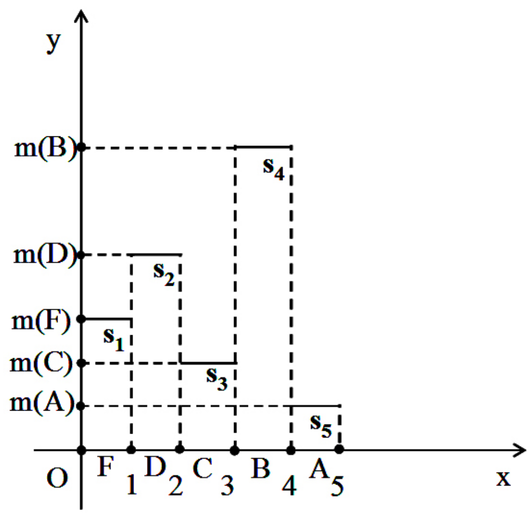

Fig. 4: The graph of the COG method

Therefore, the graph of the membership function y = m(x), takes the form of Figure 3, where the area of the level's section F contained between the graph and the OX axis is equal to the sum of the areas of the rectangles S_i, i=1, 2, 3, 4, 5.

It is straightforward then to check (e.g. see Section 3 of [2]) that in this case the formulas calculating the COG take the form:

xc= 1/2 (y_1+3y_2+5y_3+7y_4+9y_5),
yc= 1/2 (y_1^2+y_2^2+y_3^2+y_4^2+y_5^2) ...(5)

with x_1=F, x_2=D, x_3=C, x_4=B, x_5=A and y_i = $\frac{m(x_i)}{\sum_{j=1}^{5}m(x_j)}$, i = 1, 2, 3, 4, 5.

The membership function y = m(x) can be defined according to the user's goals in any compatible to the common sense way. In order to obtain assessment results comparable to the corresponding results of the GPA index, we define here y = m(x) = $\frac{n_x}{n}$ (i.e. in terms of the frequencies of Section 2). Consequently $\sum_{i=1}^{5}m(x_i)$ =1 (100%).

It is easy then to obtain (e.g. Section 3 of [2]) the following assessment criterion:

- Among two or more groups the group with the biggest x_c performs better.
- If two or more groups have the same xc $\geq$ 2.5, then the group with the higher y_c performs better.
- If two or more groups have the same x_c < 2.5, then the group with the lower y_c performs better.

In case of the ideal performance (y_5 =1 and y_i = 0 for i $\neq$ 5) the first of formulas (5) gives that x_c= $\frac{9}{2}$, while in case of the worst performance (y_1 =1 and y_i = 0 for i $\neq$ 1) it gives x_c = $\frac{1}{2}$. Therefore, values of xc greater than the mean value $\frac{10}{4}$ = 2.5 could be considered as demonstrating a more than satisfactory performance.

Due to the shape of the corresponding graph (Figure 3) the above method was named as the *Rectangular Fuzzy Assessment Model (RFAM)*.

Assessing the Student Understanding of Polar Coordinates

In an earlier work[12] we have used principles of the *APOS theory* for teaching and learning mathematics (e.g. [13, 14], etc.) to study the student understanding and proper use of the polar coordinates on the plane. One of the tools of this study was a written test performed during the academic year 2015-16 with subjects a group of 26 students of the Physics Department of the University of Neyshabur, Iran. The students had completed one month before the test the multivariable calculus course involving the concept of polar coordinates.

Table 1: The student scores in the written test

Grade	Score	Rank
A	88.5	1
B	83.5	2
B	81	3
B	80	4
B	78	5
C	72	6
C	70	7
C	69	8
C	68.5	9
C	60	10
D	57.5	11
D	57.5	12
D	57	13
D	56.75	14
F	49.5	15
F	46.5	16
F	41	17
F	39.5	18
F	33	19
F	31.5	20
F	26	21
F	23	22
F	21.5	23
F	20	24
F	17.5	25
F	9	26

The objective of the written test was to obtain a first idea about the student difficulties on the subject. Nine questions were designed for this test[12], which involved converting points and equations from polar to Cartesian coordinates and vice versa and sketching graphs of polar equations. The student results are depicted in Table 1:

The following four assessment methods are used here for evaluating the student overall performance in this test:

I) Mean value: A straightforward calculation gives that the mean value of the student scores is 51.43, demonstrating a fair (D) student mean performance.

II) GPA index: From Table 1 one finds that $n_A = 1$, $n_B = 4$, $n_C = 5$, $n_D = 4$ and $n_F = 12$. Therefore, applying formula (1) it is straightforward to find that GPA $\approx$ 1.15<2.25, which demonstrates a less than satisfactory student *quality performance.*

III) Use of TFNs: According to Definition 3.4 the mean value of the 26 in total TFNs corresponding to each student's performance is given by $M = 1/26\ [1(85, 92.5, 100)+4(75, 79.5, 84)+5(60, 67, 74)+4(50, 54.5, 59)+12(0, 24.5, 49)] \approx (34.04, 48.37, 62.69)$. Therefore, by Theorem 3.6, $x(M) = 48.37 < 50$, which demonstrates a non satisfactory (F) student mean performance

IV) RFAM: From Table 1 one finds the values $y_1 = 12/26$, $y_2 = 4/26$, $y_3 = 5/26$, $y_4 = 4/26$ $y_5 = 1/26$. Replacing these values to the first of formulas (6) it gives that. $x_c = 56/82 \approx 1.65 < 2.25$, which demonstrates a less than satisfactory student *quality performance*.

Comparison of the Assessment Methods

• On comparing the outcomes of the assessment methods I and III, one observes that while the calculation of the mean values of the student numerical scores demonstrates a fair mean performance of the student group, the calculation of the mean values of the TFNs corresponding to each student's individual performance demonstrates a non satisfactory mean performance.

The above difference is due to the different philosophy of the two methods. In fact, method I is based on the principles of the traditional bi-valued logic (Yes – No), while method III is based on FL. that characterizes the ambiguous situations with multiple values. When each student's numerical score is known, as it happens in our example, method I must be used, because it is obviously more accurate. Method III is useful in cases where each student's performance is evaluated by a qualitative grade A, B, C, D, F only and not by an exact numerical score, as it frequently happens in practice. In such cases method I is not applicable.

• The outcomes of methods II and IV are compatible to each other demonstrating a non satisfactory quality performance of the student group. In general, we can write the first of equations (5) in the form

$x_c = 1/2\ [2(y_2+ 2y_3+ 3y_4+ 4y_5) + y_1+ y_2+ y_3+ y_4+ y_5]$.

Then, by equation (2) and since $y_1 + y_2 + y_3 + y_4 + y_5 = 1$, we get that $x_c = 1/2\ (2GPA + 1)$, or finally

xc = GPA + 0.5 (6).

Therefore, if for example GPA <2 (non satisfactory performance), then (6) gives that

x_c <2.5 (also non satisfactory performance) and vice versa.

• In concluding, the combined outcomes of the four in total assessment methods that we have utilized show that the students of the group under assessment faced serious difficulties for the understanding and proper use of the polar coordinates on the plane.

Conclusion

FL, due to its property of characterizing the ambiguousv cases of real situtions with multiple values, provides reach resourses for the evaluation of such kind of cases. In the paper at hands two fuzzy assessment methods were utilized for evaluating the student understadind and proper use of the polar coordinates in the plane. The first of the above methods (use of TFNs) focuses on the student mean performance, while the second one (RFAM) focuses on the student quality performance by assigning greater coefficients to the higher scores. The applicability of these methods for the purposes of the present study was illustrated by reusing the data of a classroom application performed in an earlier work. The assessment outcomes of the above two fuzzy methods were compared to the corresponding outcomes of two traditional assessment methods of the bi-valued logic, the calculation of the mean value of the student numerical scores and of the GPA index respectively. The differences appeared between the outcomes of the fuzzy logic and the corresponding bi-valued logic assessment methods were discussed and properly justified.

The assessment methods applied in this work have a general character, which suggests that they could be also applied for evaluating a variety of other human or machine (e.g. case-based reasoning or decision-making systems with the help of computers) activities. This is one of the main targets of our future research.

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