ISSN: 2456-799X, Vol.04, No.(2) 2019, Pg. 51-64

Oriental Journal of Physical Sciences

www.orientjphysicalsciences.org

Validity of Closed Ideals In Algebras of Series of Square Analytic Functions

MUSA SIDDIG[1]* and SHAWGY HUSSIEN[2]

[1]University of Kordofan, Faculty of Science, Department of Mathematics, Sudan.
[2]Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan.

Abstract

We show the validity of a complete description of closed ideals of the algebra,

$$\mathcal{D} \cap \mathrm{lip}_{\alpha_j^2}\ , 0 < \alpha_j^2 \le \frac{1}{2}$$

where D is the algebra of series of analytic functions satisfying the Lipschitz condition of order α_j^2 obtained by.[15]

Article History

Received: 12 January 2020
Accepted: 5 February 2020

Keywords:

Banach Algebra;
Closed Ideals;
Convex Hull;
Hardy Space;
Holder Inquality.

Introduction

The Dirichlet space D consists of the sequence of square complex-valued analytic functions f_j^2 on the unit disk D with finite Dirichlet integral

$$\sum_j D(f_j^2) := \int_{\mathbb{D}} \sum_j \left|(f_j^2)'(z)\right|^2 dA(z) < +\infty ,$$

where dA(z)= $\frac{1}{\pi}(1-\epsilon)d(1-\epsilon)dt^2$

denotes the normalized area measure on D. Equipped with the pointwise algebraic operations and the series of norms

$$\sum_j \|f_j^2\|_{\mathcal{D}}^2 := \frac{1}{2\pi}\int_0^{2\pi} \sum_j \left|f_j^2(e^{it^2})\right|^2 dt^2 + D(f_j^2) = \sum_{n=0}^{\infty}\sum_j (1+n)\left|\widehat{f_j^2}(n)\right|^2,$$

D becomes a Hilbert space. For 0< α_j^2 ≤1, let lip α_j^2 be the algebra of sequence of square analytic functions f_j^2 on D that are continuous on D satisfing the Lipschitz condition of order α_j^2 on D :

$$\sum_j \left|f_j^2(z) - f_j^2(z-\epsilon)\right| = \sum_j o\left(|\epsilon|^{\alpha_j^2}\right) \qquad (|\epsilon| \to 0).$$

Note that this condition is equivalent to

CONTACT Musa Siddig muss.yousif3@gmail.com University of Kordofan, Faculty of Science, Department of Mathematics, Sudan.

Doi: http://dx.doi.org/10.13005/OJPS04.02.05

$$\sum_j |(f_j^2)'(z)| = \sum_j o((1-|z|)^{\alpha_j^2-1}) \qquad (|z| \to 1^-).$$

Then, lip α_j^2 is a Banach algebra when equipped with series of norms

$$\sum_j \|f_j^2\|_{\alpha_j^2} := \sum_j \|f_j^2\|_\infty + \sup \sum_j \{(1-|z|)^{1-\alpha_j^2}|(f_j^2)'(z)| : z \in \mathbb{D}\}.$$

Here

$$\Sigma_j \|f_j^2\|_\infty := \sup_{z\in\mathbb{D}} \Sigma_j |f_j^2(z)|.$$

Unlike as for the case when 0< α_j^2 ≤1/4, the inclusion lip α_j^2 ⊂D always holds provided that 1/4< α_j^2 ≤1. In what follows, let 0< α_j^2 ≤1/4 and define

$$\mathcal{A}_{\alpha_j^2} := \mathcal{D} \cap \text{lip}_{\alpha_j^2}$$

It is easy to check that A α_j^2 is a commutative Banach algebra when it is endowed with the pointwise algebraic operations and series of norms

$$\Sigma_j \|f_j^2\|_{\mathcal{A}_{\alpha_j^2}} := \Sigma_j \|f_j^2\|_{\alpha_j^2} + \Sigma_j D^{\frac{1}{2}}(f_j^2), \quad (f_j^2 \in \mathcal{A}_{\alpha_j^2}).$$

In order to describe the closed ideals in subalgebras of the disc algebra A(D), it is natural to make use of Nevanlinna's factorization theory. For

$$f_j^2 \in A(\mathbb{D})$$

there is a canonical factorization = $C_{f_j^2} U_{f_j^2} O_{f_j^2}$,

where $C_{f_j^2}$ is a constant, $U_{f_j^2}$ a sequence of square inner functions that is

$$\Sigma_j |U_{f_j^2}| = 1 \text{ a.e on } \mathbb{T} \text{ and } O_{f_j^2}$$

the sequence of square outer functions given by

$$\sum_j O_{f_j^2}(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \sum_j \frac{e^{i\theta^2}+z}{e^{i\theta^2}-z} \log|f_j^2(e^{i\theta^2})| d\theta^2\right\}.$$

Denote by H$^\infty$ (D) the algebra of bounded analytic functions. Note that α_j^2 has the so-called F-property [12, 2]: if $f_j^2 \in \mathcal{A}_{\alpha_j^2}$

and U is an inner function such that $f_j^2/U \in \mathcal{H}^\infty(\mathbb{D})$ then

$f_j^2/U \in \mathcal{A}_{\alpha_j^2}$ and $\Sigma_j\|f_j^2/U\|_{\mathcal{A}_{\alpha_j^2}} \le \Sigma_j C_{\alpha_j^2}\|f_j^2\|_{\mathcal{A}_{\alpha_j^2}}$ where $C_{\alpha_j^2}$ is independent of f_j^2. Korenblum[8] has described the closed ideals of the algebra H_1^2 of sequence of square analytic functions f_j^2 such that $(f_j^2)' \in H^2$ where H^2 is the Hardy space. This result has been extended to some other Banach algebras of sequence of square analytic functions, by Matheson for lip α_j^2 [9] and by Shamoyan for the algebra

$$\lambda_{z-\epsilon}^{(n)}$$

of sequence of square analytic functions f_j^2 on D such that

$$\Sigma_j |f_j^2)^{(n)}((z-2\epsilon)_1) - (f_j^2)^{(n)}((z-2\epsilon)_1 - \epsilon)| = o(\omega(|\epsilon|)) \text{ as } |\epsilon| \to 0$$

where n is a non negative integer and ω an arbitrary nonnegative non decreasing subadditive function on (0,+∞).[11] Shirokov[13, 12] had given a complete description of closed ideals for Besov algebras

$$AB_{1+\epsilon,1+\epsilon}^{\left(\frac{1}{2}+\epsilon\right)}$$

of sequence of square analytic functions and particularly for the case ϵ >0.

$$AB_{2,2}^{\left(\frac{1}{2}+\epsilon\right)} = \left\{(f_j^2 \in A(\mathbb{D}): \sum_{n\ge 0}\sum_j |\widehat{f_j^2}(n)|^2 (1+n)^{(1+2\epsilon)} < \infty\right\}$$

Note that the case of $AB_{2,2}^{\frac{1}{2}} = A(\mathbb{D}) \cap \mathcal{D}$

the problem of description of closed ideals appears to be much more difficult (see[6, 4]). Brahim Bouya[15] described the structure of the closed ideals of the Banach algebras $\mathcal{A}_{\alpha_j^2}$. More precisely he proved that these ideals are standard in the sense of the Beurling-Rudin characterization of the closed ideals in the disc algebra[7], we show the general validation following[15]

Theorem

If I is closed ideal of $\mathcal{A}_{\alpha_j^2}$, then

$$\mathfrak{I} = \left\{f_j^2 \in \mathcal{A}_{\alpha_j^2}: (f_j^2)_{\backslash E_\mathfrak{I}} = 0 \text{ and } f_j^2/U_\mathfrak{I} \in \mathcal{H}^\infty(\mathbb{D})\right\},$$

where

$E_{\mathfrak{T}} := \{z \in \mathbb{T} : \sum_j f_j^2(z) = 0, \forall f_j^2 \in \mathfrak{T}\}$

and

$U_{\mathfrak{T}}$ is greatest common divisor of the inner parts of the non-zero functions in $\mathfrak{T}$.

Such characterization of closed ideals can be reduced further to a problem of approximation of outer functions using the Beurling– Carleman–Domar resolvent method. Define d(ξ,E) to be the distance from ξ ϵ T to the set E ⊂ T. Suppose that $\mathfrak{T}$ is a closed ideal in $\mathcal{A}_{\alpha_j^2}$ such that $U_{\mathfrak{T}}$ =1.

We have $Z_{\mathfrak{T}} = E_{\mathfrak{T}}$, where

$$Z_{\mathfrak{T}} := \left\{ z \in \mathbb{D}: \sum_j f_j^2(z) = 0 , \quad \forall f_j^2 \in \mathfrak{T} \right\}.$$

Next, for $f_j^2 \in \mathcal{A}_{\alpha_j^2}$ such that

$$\sum_j |f_j^2(\xi)| \le \sum_j C d(\xi, E_{\mathfrak{T}})^{M_{\alpha_j^2}} \qquad (\xi \in \mathbb{T}),$$

where $M_{\alpha_j^2}$ is a positive constant depending only on,

$\mathcal{A}_{\alpha_j^2}$ we have $f_j^2 \in \mathfrak{T}$ (see section 3 for more precisions). Now, to show Theorem (1.1) we need Theorem (1.2) below, which states that every function in $\mathcal{A}_{\alpha_j^2}$ \{0} can be approximated in $\mathcal{A}_{\alpha_j^2}$

by functions with boundary zeros of arbitrary high order.

Theorem
Let f_j^2 be a function in $\mathcal{A}_{\alpha_j^2}$ \ {0} and let $\epsilon \ge 0$.

There exists a sequence of functions

$\{(g_j)_n\}_{n=1}^{\infty} \subset A(\mathbb{D})$ such that

For all n ϵ N, we have $\sum_j (f_j^2)_n = \sum_j f_j^2 (g_j^2)_n \in \mathcal{A}_{\alpha_j^2}$ and

$$Lim_{n\to\infty} \sum_j \left\| (f_j^2)_n - f_j^2 \right\|_{\mathcal{A}_{\alpha_j^2}} = 0$$

$$\sum_j \left|(g_j^2)(\xi)\right| \le \sum_j C_n d^{1+\epsilon}\left(\xi, E_{f_j^2}\right) \quad (\xi \in T)$$

where

$$E_{f_j^2} := \{\xi \in T : \textstyle\sum_j f_j^2(\xi) = 0\}$$

To show this Theorem, we give a refinement of the classical Korenblum approximation theory./[8, 9, 11, 13, 12]

Main result on approximation of functions in $\mathcal{A}_{\alpha_i^2}$

Let $f_j^2 \in \mathcal{A}_{\alpha_i^2}$ and let $\{\gamma_n := (a_n, (a+\epsilon)_n)\}_{n\ge 0}$

be the countable collection of the (disjoint open) arcs of $\mathbb{T} \setminus E_{f_i^2}$.

We can suppose that the arc lengths of γ_n are less than 1/2. In what follows, we denote by Γ the union of a family of arcs γ_n. Define

$$\sum (f_j^2)_\Gamma (z) := \exp\left\{ \frac{1}{2\pi} \int_\Gamma \sum \frac{e^{i\theta^2} + z}{e^{i\theta^2} - z} \log\left|f_j^2(e^{i\theta^2})\right| d\theta^2 \right\}.$$

The difficult part in the proof of Theorem (1.2) is to establish the following

Theorem
Let Let $f_j^2 \in \mathcal{A}_{\alpha_j^2} \setminus \{0\}$

be an outer function such that $\sum_j \left\|f_j^2\right\|_{\mathcal{A}_{\alpha_j^2}} \le 1$

and let ϵ ≥1 and ϵ > 0. Then we have

$$f_j^{2(1+\epsilon)} (f_j)_\Gamma^{2(1+\epsilon)} \in \mathcal{A}_{\alpha_j^2} \text{ and } \sup_\Gamma \sum_j \left\| f_j^{2(1+\epsilon)} (f_j)_\Gamma^{2(1+\epsilon)} \right\|_{\mathcal{A}_{\alpha_j^2}} \le C_{1+\epsilon,1+\epsilon} \;, \qquad ...(1)$$

where $C_{1}+\epsilon,1+\epsilon$) is a positive constant independent of Γ.

Remark
For a set S ⊂ A(D), we denote by *co(S)* the convex hull of S consisting of the intersection of all convex sets that contain S. Set $\Gamma_n = \bigcup_{\epsilon\ge 0} \gamma_{n+\epsilon}$

and let f_j^2 be as in the Theorem (2.1) It is clear that the sequence

$$(f_j^{2(1+\epsilon)} (f_j)_{\Gamma_n}^{2(1+\epsilon)})$$

converges uniformly on compact subsets of D to $f_j^{2(1+\epsilon)}$.

We use (2.1) to deduce, by the Hilbertian structure of D, that there is a sequence $(h_j^2)_n \in$

$$co(\{f_j^{2(1+\epsilon)}(f_j)_{\Gamma_{1+\epsilon}}^{2(1+\epsilon)}\}_{\epsilon=0}^{\infty})$$

converging to $f_j^{2(1+\epsilon)}$ in D. Also, by [9, section4], we obtain that

$(h_j^2)_n$ converges to $f_j^{2(1+\epsilon)}$ in lip α_j^2, for

sufficiently large (1+ϵ) (in fact, we can show that this result remains true for every ϵ ≥ 0). Therefore

$$\sum_j \left\| (h_j^2)_n - f_j^{2(1+\epsilon)} \right\|_{\mathcal{A}_{\alpha_j^2}} \to 0, \quad \text{as } n \to \infty$$

Define J(F) to be the closed ideal of all functions in

$\mathcal{A}_{\alpha_j^2}$ that vanish on. $F \subset \overline{\mathbb{D}}$ In the proof of Theorem (1.2), we need the following classical lemma (see[15]), see for instance [[9], Lemma [4]] and [[8], Lemma [24]].

Lemma

Let $f_j^2 \in \mathcal{A}_{\alpha_j^2}$ and E' be a finite subset of T such that

$$\sum_j f_j^2 | E' = 0. Let\ \epsilon \geq 0$$

be given. For every ϵ > 0 there is an outer function F in J(E') such that

$$\sum_j \|F f_j^2 - f_j^2\|_{\mathcal{A}_{\alpha_j^2}} \leq \varepsilon,$$

$$|F(\xi)| \leq C d^{1+\epsilon}(\xi, E') \quad (\xi \in \mathbb{T})$$

Proof of Theorem

Now, we can deduce the proof of Theorem (1.2) by using Theorem (2.1) and Lemma (2.3) Indeed, let be f_j^2 a sequence of functions in $\mathcal{A}_{\alpha_j^2} \backslash \{0\}$

such that $\sum_j \|f_j^2\|_{\mathcal{A}_{\alpha_j^2}} \leq 1$

and let ϵ > 0. For ϵ ≥ 0 we have

$$\sum_j \left(f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} - f_j^2\right)' = \sum_j \left(O_{f_j^2}^{\frac{1}{1+\epsilon}} - f_j^2\right)(f_j^2)' + \sum_j \tfrac{1}{1+\epsilon} U_{f_j^2} O_{f_j^2}^{\frac{1}{1+\epsilon}} O'_{f_j^2}$$

The F-property of α_j^2 implies that $O_{f_j^2} \in \mathcal{A}_{\alpha_j^2}$.

Then, there exists $\eta_0 \in \mathbb{N}$ such that

$$\sum_j \left\| f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} - f_j^2 \right\|_{\mathcal{A}_{\alpha_j^2}} < \frac{\epsilon}{3} \qquad (\epsilon \geq 0)$$

Set $\Gamma_n = \cup_{1+\epsilon \geq n} \gamma_{1+\epsilon}$ and $\alpha_j^2 \leq 1$ for a given

ϵ ≥ 0. By Remark (2.2) applied to $O_{f_j^2}$ (with ϵ ≥ 0),

there is a sequence $k_{n,1+\epsilon} \in co\left(\left\{(f_j)_{\Gamma_{1+\epsilon}}^{1+\epsilon}\right\}_{\epsilon=0}^{\infty}\right)$ such that

$$\sum_j \left\| O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}} k_{n,1+\epsilon} - O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}} \right\|_{\mathcal{A}_{\alpha^2}} < \frac{1}{1+\epsilon} \quad (n \in \mathbb{N}, \ \epsilon \geq 0)$$

It is clear that

$$\sum_j \left\| O_{f_j^2}^{\frac{1}{1+\epsilon}} (f_j)_{\Gamma_n}^{2(1+\epsilon)} - O_{f_j^2}^{\frac{1}{1+\epsilon}} \right\|_{\infty} \longrightarrow 0 \qquad (n \longrightarrow +\infty)$$

Then for every ϵ > 0 we get

$$\sum_j \left\| O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{n,1+\epsilon} - O_{f_j^2}^{\frac{1}{1+\epsilon}} \right\|_{\infty} \longrightarrow 0 \qquad (n \longrightarrow +\infty)$$

So, there is a sequence

$$k_{1+\epsilon} \in co\left(\left\{(f_j)_{\Gamma_{1+\epsilon}}^{2(1+\epsilon)}\right\}_0^{\infty}\right)$$

such that

$$\begin{cases} \sum_j \left\| O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}} k_{1+\epsilon} - O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}} \right\|_{\mathcal{A}_{\alpha_j^2}} \leq \frac{1}{1+\epsilon} & (\epsilon \geq 0), \\ \sum_j \left\| O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - O_{f_j^2}^{\frac{1}{1+\epsilon}} \right\|_{\infty} \leq \frac{1}{1+\epsilon} & (\epsilon \geq 0). \end{cases}$$

We have

$$\sum_j \left(f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}}\right)' =$$
$$\sum_j \left((f_j^2)' - U_{f_j^2} O'_{f_j^2}\right)\left(O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - O_{f_j^2}^{\frac{1}{1+\epsilon}}\right) + \sum_j \left(U_{f_j^2} O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}} k_{1+\epsilon} - O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}}\right)'$$

Since

$$\sum_j \left\| O_{f_j^2} \right\|_{\mathcal{A}_{\alpha_j^2}} \leq \sum_j C_{\alpha_j^2} \|f_j^2\|_{\alpha_j^2} \leq \sum_j C_{\alpha_j^2}$$

we obtain

$$\sum_j \left\| f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} \right\|_{\mathcal{A}_{\alpha_j^2}} \sum_j \left\| f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} \right\|_\infty$$
$$+ \ sup_{z\in\mathbb{D}} \left\{ \sum_j (1-|z|)^{1-\alpha_j^2} \left| \left(f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} \right)' (z) \right| \right\} +$$
$$\sum_j D^{\frac{1}{2}} \left(f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} \right) \sum_j C_{\alpha_j^2} \|f_j^2\|_{\alpha_j^2} \left\| O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - O_{f_j^2}^{\frac{1}{1+\epsilon}} \right\|_\infty$$
$$\leq \sum_j \left\| f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} \right\|_\infty + + sup_{z\in\mathbb{D}} \left\{ \sum_j (1-|z|)^{1-\alpha_j^2} \right\}$$
$$+ C \sum_j \left\| O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - O_{f_j^2}^{\frac{1}{1+\epsilon}} \right\|_\infty + \sum_j D^{\frac{1}{2}}(f_j^2) + C D^{\frac{1}{2}} \sum_j \left(O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}} k_{1+\epsilon} - O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}} \right)$$
$$\leq \sum_j C_{\alpha_j^2} \left\| O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - O_{f_j^2}^{\frac{1}{1+\epsilon}} \right\|_\infty + C \sum_j \left\| O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}} k_{1+\epsilon} - O_{f_j^2}^{\frac{2+\epsilon}{1+\epsilon}} \right\|_{\mathcal{A}_{\alpha_j^2}} \leq \sum_j \frac{C_{\alpha_j^2}}{1+\epsilon}$$

Then, fix $\epsilon > 0$ such that

$$\sum_j \left\| f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} \right\|_{\mathcal{A}_{\alpha_j^2}} < \epsilon/3 \qquad (\epsilon \geq 0).$$

We have $k_{1+\epsilon} = \sum_{i\leq j_{1+\epsilon}} \sum_j c_i f_{\Gamma_i}^{2(1+\epsilon)}$

where

$\sum_{i\leq j_{1+\epsilon}} c_i = 1$. Set $E'_{1+\epsilon} = \cup_{i\leq j_{1+\epsilon}} \partial\gamma_i$

Using Lemma (2.3), we obtain an outer function

$F_{1+\epsilon} \in \mathcal{J}(E'_{1+\epsilon})$ such that

$|F_{1+\epsilon}(\zeta)| \leq C_{1+\epsilon} d^{1+\epsilon}(\zeta, E'_{1+\epsilon})$ and

$$\sum_j \left\| f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} F_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} \right\|_{\mathcal{A}_{\alpha_j^2}} < \frac{1}{1+\epsilon} \ ,(\epsilon \geq 1)$$

Then fix $\epsilon > 0$ such that

$$\sum_j \left\| f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} F_{1+\epsilon} - f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} \right\|_{\mathcal{A}_{\alpha_j^2}} < \epsilon/3 \quad (\epsilon \geq 0)$$

Consequently we obtain

$$\sum_j \left\| f_j^2 O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} F_{1+\epsilon} - f_j^2 \right\|_{\mathcal{A}_{\alpha_j^2}} < \epsilon \qquad (\epsilon \geq 0)$$

It is not hard to see that

$$\sum_j \left| O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} F_{1+\epsilon}(\xi) \right| \leq \sum_j C_{1+\epsilon} d^{1+\epsilon} \left(\xi, E_{f_j^2} \right) \qquad (\xi \in \mathbb{T})$$

Therefore

$$\sum_j (g_j^2)_{1+\epsilon} = \sum_j O_{f_j^2}^{\frac{1}{1+\epsilon}} k_{1+\epsilon} F_{1+\epsilon}$$

is the desired series of sequence, which completes the proof of Theorem (1.2).

Beurling – Carleman – Domar resolvent methed

Since $\mathcal{A}_{\alpha_j^2} \subset \mathrm{lip}_{\alpha_j^2}$, then for all $f_j^2 \in \mathcal{A}_{\alpha_j^2}$, $E_{f_j^2}$

satisfies the Carleson condition

$$\int_{\mathbb{T}} \sum_j \log \frac{1}{d(e^{it^2}, E_{f_j^2})} dt^2 < +\infty$$

For $f_j^2 \in \mathcal{A}_{\alpha_j^2}$,

we denote by $B_{f_j^2}$

the Blashke product with zeros $Z_{f_j^2} \backslash E_{f_j^2}$,

where $Z_{f_j^2} := \{z \in \overline{\mathbb{D}} : \sum_j f_j^2(z) = 0\}$.

We begin with following lemma (see[15]).

Lemma

Let $\mathfrak{I}$ be a closed ideal of $\mathcal{A}_{\alpha_j^2}$. Define $B_{\mathfrak{I}}$

to be the Blashke product with zeros $Z_{\mathfrak{I}} \backslash E_{\mathfrak{I}}$

There is a sequence of functions $f_j^2 \in \mathfrak{I}$

such that $B_{f_j^2} = B_{\mathfrak{I}}$.

Proof

Let $g_j^2 \in \mathfrak{T}$ and let B_n be the Blashke product with zeros

$Z_{g_j^2} \cap \mathbb{D}_n$, where $(g_j^2)_n$

$\mathbb{D}_n := \{z \in \mathbb{D} : |z| < \frac{n-1}{n}, \ n \in \mathbb{N}\}$,

where $K_n = B_n/I_n$ and I_n is the Blashke product with zeros

$Z_{g_j^2} \cap \mathbb{D}_n$. We have $(g_j^2)_n \in I$ for every

n. Indeed, fix $n \in \mathbb{N}$.

It is permissible to assume that Z_{K_n} consists of a single point, say

$Z_{K_n} = \{z - \epsilon\}$. Let $\pi : \mathcal{A}_{\alpha_j^2} \to \mathcal{A}_{\alpha_j^2}/\mathfrak{T}$

be the canonical quotient map. First suppose

$(z - \epsilon) \notin Z_{\mathfrak{T}}$ then $\pi(K_n)$ is invertible in

$\mathcal{A}_{\alpha_j^2}/\mathfrak{T}$. It follows that

$$\sum_j \pi(g_j^2)_n = \sum_j \pi(g_j^2)\pi^{-1}(K_n) = 0$$

hence $(g_j^2)_n \in \mathfrak{T}$.

If $(z - \epsilon) \in Z_{\mathfrak{T}}$, we consider the following ideal

$$\mathcal{J}_{z-\epsilon} := \{f_j^2 \in \mathcal{A}_{\alpha_j^2} : f_j^2 I_n \in \mathfrak{T}\}.$$

It is clear that $\mathcal{J}_{z-\epsilon}$ is closed. Since

$(z - \epsilon) \notin Z_{\mathcal{J}_{z-\epsilon}}$, it follows that K_n is invertible in the quotient algebra $\mathcal{A}_{\alpha_j^2}/\mathcal{J}_{z-\epsilon}$ and so

$g_j^2/(I_n K_n) \in \mathcal{J}_{z-\epsilon}$ Hence $(g_j^2)_n \in \mathfrak{T}$

It is clear that $(g_j^2)_n$

converges uniformly on compact subsets of D to

$$\sum_j f_j^2 = \sum_J (g_j^2/B_{g_j^2}) B_{\mathfrak{T}}$$

and we have $\sum_J B_{f_j^2} = B_{\mathfrak{T}}$.

In the sequel we prove that .

If we obtain

$$\sum_j \left|\left((g_j^2)_n\right)'(z)\right| \le \sum_j o\left(\frac{1}{\epsilon^{1-\alpha_j^2}}\right) \qquad (z \in \mathbb{D}),$$

uniformly with respect to n, we can deduce by using [9, Lemma 1] that

$$\lim_{n\to+\infty} \sum_j \left\|(g_j^2)_n - f_j^2\right\|_{\alpha_j^2} = 0 .$$

Indeed, by the Cauchy integral formula

$$(z - 2 = \frac{1}{2\pi i}\int_{\mathbb{T}} \sum_j \frac{\left(g_j^2(z-2\epsilon) - g_j^2(z/|z|)\right)\overline{K_n(z-2\epsilon)}}{4\epsilon^2} d(z-2\epsilon) \ (z \in \mathbb{D})$$

Then, for $z = (1-\epsilon)e^{i\theta^2} \in \mathbb{D}$

$$\sum_j \left((g_j^2)_n\right)'(z) \le \frac{\|K_n\|_\infty}{2\pi}\int_{\mathbb{T}} \sum_j \frac{|g_j^2(z-2\epsilon) - g_j^2(z/|z|)|}{4|\epsilon|^2} |d(z-2\epsilon)|$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_j \frac{|g_j^2(e^{i(t^2+\theta^2)}) - g_j^2(e^{i\theta^2})|}{(2\epsilon-1)\cos t^2 + (1-\epsilon)^2} dt^2 .$$

For all $\epsilon > 0$, there is $\epsilon > 0$ such that if $|t^2| \le n$, we have

$$\sum_j |g_j^2(e^{i(t^2+\theta^2)}) - g_j^2(e^{i\theta^2})| \le \sum_j \varepsilon |t^2|^{\alpha_j^2} \quad (\theta^2 \in [-\pi, +\pi])$$

Then

$$\int_{-\pi}^{\pi}\sum_j \frac{|g_j^2(e^{i(t^2+\theta^2)}) - g_j^2(e^{i\theta^2})|}{(2\epsilon-1)\cos t^2 + (1-\epsilon)^2} dt^2$$
$$\leq \varepsilon \int_{|t^2|\leq\eta} \sum_j \frac{|t^2|^{\alpha_j^2}}{\epsilon^2 + 4(1-\epsilon)t^2/\pi^2} dt^2$$
$$+ \sum_j \|g_j^2\|_{\alpha_j^2} \int_{|t^2|\leq\eta} \sum_j \frac{|t^2|^{\alpha_j^2}}{\epsilon^2 + 4(1-\epsilon)t^2/\pi^2} dt^2$$
$$\leq \sum_j \frac{\varepsilon}{(1-\epsilon)^{\frac{1+\alpha_j^2}{2}}\epsilon^{1-\alpha_j^2}} \int_0^{+\infty} \sum_j \frac{u^{\alpha_j^2}}{1+(2u/\pi)^2} du$$
$$+ \sum_j \frac{\|g_j^2\|_{\alpha_j^2}}{(1-\epsilon)^{\frac{1+\alpha_j^2}{2}}\epsilon^{1-\alpha_j^2}} \int_{|u|\geq\frac{\eta\sqrt{1-\epsilon}}{\epsilon}} \sum_j \frac{u^{\alpha_j^2}}{1+(2u/\pi)^2} du$$
$$\leq \sum_j \varepsilon O\left(\frac{1}{\epsilon^{1-\alpha_j^2}}\right) + \sum_j \|g_j^2\|_{\alpha_j^2} O\left(\frac{1}{\epsilon^{1-\alpha_j^2}}\right).$$

we obtain

$$b\int_{-\pi}^{\pi}\sum_j \frac{|g_j^2(e^{i(t^2+\theta^2)}) - g_j^2(e^{i\theta^2})|}{(2\epsilon-1)\cos t^2 + (1-\epsilon)^2} dt^2 \leq \sum_j \|g_j^2\|_{\alpha_j^2} O\left(\frac{1}{\epsilon^{1-\alpha_j^2}}\right) \quad ...(2)$$

Consequently

$$\sum_j \left|\left((g_j^2)_n\right)'(z)\right| \leq \sum_j \|g_j^2\|_{\alpha_j^2} O\left(\frac{1}{\epsilon^{1-\alpha_j^2}}\right) \quad (z \in \mathbb{D})$$

By the F-property of $\mathcal{A}_{\alpha_j^2}$, we have

$$\Sigma_j \left\|(g_j^2)_n\right\| \leq \Sigma_j C_{\alpha_j^2} \left\|(g_j^2)_n\right\|_{\mathcal{A}_{\alpha_j^2}}.$$

Using the Hilbertian structure of D, we deduce that there is a sequence $(h_j^2)_n \in co(\{(g_j^2)_k\}_{k=n}^{\infty})$

converging to f_j^2 in D. It is clear that

$(h_j^2)_n \in \mathfrak{T}$ and $\lim_{n\to+\infty} \Sigma_j \left\|(h_j^2)_n - f_j^2\right\|_{\alpha_j^2} = 0$

Then

$$\lim_{n\to+\infty} \Sigma_j \left\|(h_j^2)_n - f_j^2\right\|_{\mathcal{A}_{\alpha_j^2}} = 0$$

Thus

$f_j^2 \in \mathfrak{T}$

This completes the proof of the lemma.

We can see that

$$\Sigma_j \left\|(g_j^2)_n\right\|_{\alpha_j^2} O\left(\frac{1}{\epsilon^{1-\alpha_j^2}}\right) = \Sigma_j O\left(\frac{1}{\epsilon^{1-\alpha_j^2}}\right).$$

As a consequence of Theorem (1.2), we can show Theorem (1.1) and deduce that each closed ideal of

$\mathcal{A}_{\alpha_j^2}$

is standard. For the sake of completeness, we sketch here the proof,(see[15]).

Proof of Theorem

Define γ on D by $\gamma(z)=z$ and let π :

$\mathcal{A}_{\alpha_j^2} \to \mathcal{A}_{\alpha_j^2}/\mathfrak{T}$ be the canonical quotient map.

Also, let $f_j^2 \in \mathcal{J}(E_{\mathfrak{T}})$ be such that

$f_j^2/U_{\mathfrak{T}} \in \mathcal{H}^{\infty}(\mathbb{D})$ and $(f_j^2)_n$

be the sequence in Theorem (1.2) associated to

f_j^2 with $\epsilon \geq 2$. More exactly, we have

$\Sigma_j (f_j^2)_n = \Sigma_j f_j^2 (g_j^2)_n$, where

$$\Sigma_j |(g_j^2)_n(\xi)| \leq \Sigma_j d^3(\xi, E_{f_j^2}) \leq d^3(\xi, E_{\mathfrak{T}})$$

Define

$$\sum_j L_\lambda(f_j^2)(z) := \begin{cases} \sum_j \frac{f_j^2(z) - f_j^2(\lambda)}{z-\lambda} & \text{if } z \neq \lambda, \\ \sum_j (f_j^2)'(\lambda) & \text{if } z = \lambda. \end{cases}$$

Then

$$\sum_j \pi(f_j^2)(\pi(\gamma)-\lambda)^{-1} = \sum_j f_j^2(\lambda)(\pi(\gamma)-\lambda)^{-1} + \sum_j \pi\left(L_\lambda(f_j^2)\right).$$

It is clear that$(\pi(\gamma)-\lambda)^{-1}$ is an analytic function on $\mathbb{C}\backslash Z_{\mathfrak{I}}$.

Note that the multiplicity of the pole

$z_0 \in Z_{\mathfrak{I}} \cap \mathbb{D}$ of $(\pi(\gamma)-\lambda)^{-1}$

is equal to the multiplicity of the zero z_0 of $U_{\mathfrak{I}}$

Since $U_{\mathfrak{I}}$ divides f_j^2, then according to[3] we can deduce that

$$\sum_j \pi(f_j^2)(\pi(\gamma)-\lambda)^{-1}$$

is a series of square analytic functions on

$\mathbb{C}\backslash E_{\mathfrak{I}}$. Let $|\lambda| > 1$, we have

$$\sum_j \|\pi(f_j^2)(\pi(\gamma)-\lambda)^{-1}\|_{\mathcal{A}_{\alpha_j^2}} \leq \sum_j \|f_j^2\|_{\mathcal{A}_{\alpha_j^2}} \sum_{n=0}^{\infty}\sum_j \|\gamma^n\|_{\mathcal{A}_{\alpha_j^2}} |\lambda|^{-n-1} \leq \sum_j \|f_j^2\|_{\mathcal{A}_{\alpha_j^2}} \frac{C}{(|\lambda|-1)^{\frac{3}{2}}}. \quad ...(4)$$

By Lemma (3.1), there is

$g_j^2 \in \mathfrak{I}$ such that

$B_{g_j^2} = B_{\mathfrak{I}}$. Let $k = \sum_j f_j^2(g_j^2/B_{g_j^2})$.

Then,

$k = \sum_j (f_j^2/B_{\mathfrak{I}})g_j^2 \in \mathfrak{I}$ and for $|\lambda| < 1$

we have

$k(\lambda)(\pi(\gamma)-\lambda)^{-1} = -\pi(L_\lambda(k))$.

Therefore

$$\sum_j \|\pi(f_j^2)(\pi(\gamma)-\lambda)^{-1}\|_{\mathcal{A}_{\alpha_j^2}} \leq \sum_j |f_j^2(\lambda)| \|(\pi(\gamma)-\lambda)^{-1}\|_{\mathcal{A}_{\alpha_j^2}} + \sum_j \|L_\lambda(f_j^2)\|_{\mathcal{A}_{\alpha_j^2}} \leq \sum_j \frac{\|L_\lambda(k)\|_{\mathcal{A}_{\alpha_j^2}}}{\left|\frac{g_j^2}{B_{g_j^2}}\right|(\lambda)} + \leq \sum_j \frac{C(f_j^2,k)}{(1-|\lambda|)\left|g_j^2/B_{g_j^2}\right|(\lambda)} \leq \sum_j C(f_j^2,k)e^{\frac{c}{1-|\lambda|}} \quad (|\lambda|<1). \quad ...(5)$$

We use [14, Lemmas 5.8 and 5.9] to deduce

$$\sum_j \|\pi(f_j^2)(\pi(\gamma)-\xi)^{-1}\| \leq \sum_j \frac{C(f_j^2,k)}{d(\xi,E_{\mathfrak{I}})^3} \qquad (1 \leq |\xi| \leq 2,\ \xi \notin E_{\mathfrak{I}})$$

Then, we obtain

$$\xi \longmapsto \sum_j |((g_j^2)_n)(\xi)| \|\pi(f_j^2)(\pi(\gamma)-\xi)^{-1}\| \in L^\infty(\mathbb{T})$$

With a simple calculation as in [5, Lemma 2.4], we can deduce that

$$\sum_j \pi((f_j^2)_n) = \frac{1}{2\pi i}\int_{\mathbb{T}} \sum_j ((g_j^2)_n)(\xi)(\pi(\gamma)-\xi)^{-1} d\xi$$

Denote

$$\mathfrak{I}_{U_{\mathfrak{I}}}^{\infty}(E_{\mathfrak{I}}) := \left\{h_j^2 \in A(\mathbb{D}) \colon (h_j^2)_{\backslash E_{\mathfrak{I}}} = 0 \text{ and } h_j^2 / U_{\mathfrak{I}} \in A(\mathbb{D})\right\}$$

From [7, p. 81], we know that $\mathfrak{I}_{U_{\mathfrak{I}}}^{\infty}(E_{\mathfrak{I}})$

has an approximate identity $(e_{1+\epsilon})_{\epsilon\geq 0} \in \mathfrak{I}$ such that

$\|e_{1+\epsilon}\|_\infty \leq 1$. $\mathfrak{I}$ is dense in $\mathfrak{I}_{U_{\mathfrak{I}}}^{\infty}(E_{\mathfrak{I}})$

with respect to the sup norm $\|\cdot\|_\infty$, so there exists

$(u_{1+\epsilon})_{\epsilon\geq 0} \in \mathfrak{I}$ with $\|u_{1+\epsilon}\|_\infty \leq 1$ and

$\lim_{1+\epsilon\to\infty} u_{1+\epsilon}(\xi) = 1$ for $\xi \in \mathbb{T}\backslash E_{\mathfrak{I}}$. Therefore

$\sum_j \pi((f_j^2)_n) = \sum_j \pi\left((f_j^2)_n - (f_j^2)_n u_{1+\epsilon}\right) \to 0$ as $\epsilon \to \infty$.

as $\epsilon \to \infty$ Then

$(f_j^2)_n \in \mathfrak{I}$ and $f_j^2 \in \mathfrak{I}$.

Note that: if

$\lim_{n\to\infty}\sum_j|(g_j^2)_n(\xi)| = \sum_j|(g_j^2)|\,|\xi|$

then,

$\sum_j cd^{1+\epsilon}(\xi, E_{f_j^2}) = \sum_j d^3(\xi, E_{f_j^2})$

Proof of Theorem

The proof of Theorem (2.1) is based on a series of lemmas. In what follows, $C_{1+\epsilon}$ will denote a positive number that depends only on 1+ϵ, not necessarily the same at each occurrence. For an open subset Δ of D, we put

$$\sum_j \|((h_j^2)'\|^2_{L^2(\Delta)} := \int_\Delta \sum_j |(f_j^2)'(z)|^2 dA(z)$$

We begin with the following key lemma (see15).

Lemma (4.1)

Let $f_j^2 \in \mathcal{A}_{f_j^2}$ be such that

and let $\epsilon > 0$ be given. Then

$$\int_\gamma \sum_j \frac{|f_j^2(e^{it^2})|^{2(1+\epsilon)}}{d(e^{it^2})} dt^2 \le \sum_j C_{1+\epsilon}\|(f_j^2)'\|^2_{L^2(\gamma)}$$

where $a, a+\epsilon \in E_{\mathfrak{X}}, \gamma = (a, a+\epsilon) \subset \mathbb{T}\backslash E_{f_j^2}$,

$d(z) := \min\{|z-a|, |z-(a+\epsilon)|\}$ and

$\Delta_\gamma := \{z \in D: z/|z| \in \gamma\}$

Proof

Let $e^{it^2} \in \gamma$ and define $z_{t^2} := (1 - d(e^{it^2}))e^{it^2}$

Since $|\gamma|<1/2$, we obtain $|z_{t^2}| > 1/2$.

We have $\sum_j\|f_j^2\|_{\mathcal{A}_{\alpha_j^2}} \le 1$

$$\sum_j |f_j^2(e^{it^2})|^{2(1+\epsilon)}$$
$$\le \sum_j 2^{2\epsilon+1}(|f_j^2(e^{it^2}) - f_j^2(z_{t^2})|^{2(1+\epsilon)} + |f_j^2(z_{t^2})|^{2(1+\epsilon)}). \qquad ...(6)$$

By H¨older's inequality combined with the fact that $\sum_j\|f_j^2\|_\infty \le \sum_j\|f_j^2\|_{\mathcal{A}_{\alpha_j^2}} \le 1$, we get

$$\sum_j |f_j^2(e^{it^2}) - f_j^2(z_{t^2})|^{2(1+\epsilon)} = \sum_j |f_j^2(e^{it^2}) - f_j^2(z_{t^2})|^{2\epsilon}|f_j^2(e^{it^2}) - f_j^2(z_{t^2})|^2$$
$$\le 2^{2\epsilon}(1-|z_{t^2}|)\int_{|z_{t^2}|}^1 \sum_j |(f_j^2)'((1-\epsilon)e^{it^2})|^2 (1-\epsilon)d(1-\epsilon)$$
$$\le 2^{2\epsilon+1}d(e^{it^2})\int_0^1 \sum_j |(f_j^2)'((1-\epsilon)e^{it^2})|^2 (1-\epsilon)d(1-\epsilon).$$

Hence

$$\int_\gamma \sum_j \frac{|f_j^2(e^{it^2}) - f_j^2(z_{t^2})|^{2(1+\epsilon)}}{d(e^{it^2})} dt^2$$
$$\le 2^{(2\epsilon+1)}\int_\gamma\int_0^1 \sum_j |(f_j^2)'(re^{it^2})|^2 (1-\epsilon)d(1-\epsilon)dt^2$$
$$\le \sum_j 2^{(2\epsilon+1)}\pi\|(f_j^2)'\|^2_{L^2(\Delta_\gamma)}. \qquad ...(7)$$

Since $d(e^{it^2}) \le 1/2$, we obtain

$\frac{d(e^{it^2})}{\sqrt{2}} \le d(z_{t^2}) \le \sqrt{2}d(e^{it^2})$.Put $d(z_{t^2}) = |z_{t^2} - \xi|$

and note that either $\xi = a$ or $\xi = a + \epsilon$. Let

$$z_{t^2}(u) = (1-u)z_{t^2} + u\xi \qquad (0 \le u \le 1)$$

With a simple calculation, we can prove that for all $e^{it^2} \in \gamma$

and for all u ,0≤u≤1, we have

$|z_{t^2}(u) - w| > \frac{1}{2}(1-u)d(e^{it^2})$ $(w \in \partial\Delta_\gamma)$, where $\partial\Delta_\gamma$

is the boundary of $\Delta\gamma$. Then

$\mathbb{D}_{t^2,u} := \{z \in \mathbb{D}: |z - z_{t^2}t^2(u)| \le \frac{1}{2}(1-u)d(e^{it^2})\} \subset \Delta_\gamma,$

for all $e^{it^2} \in \gamma$

and for all u,0≤u≤1. Since $\sum_j|(f_j^2)'(z)|$

is a series of subharmonic on D, it follows that

$$\sum_j |(f_j^2)'(z_{t^2}(u))| \le \frac{4}{\pi(1-u)^2 d^2(e^{it^2})}\int_{\mathbb{D}_{t,u}} \sum_j |(f_j^2)'(z)| dA(z)$$
$$\le \frac{2}{\pi^{\frac{1}{2}}(1-u)\,d\,(e^{it^2})}\sum_j \|(f_j^2)'\|_{L^2(\Delta_\gamma)}.$$

Set $\varepsilon_{(1+\epsilon)} = 2\alpha_j^2\epsilon$.

We have

$$\sum_j \left|f_j^{2(1+\epsilon)}(z_{t^2})\right|^2 = \sum_j \left|f_j^{2(1+\epsilon)}(z_{t^2}) - f_j^{2(1+\epsilon)}(\xi)\right|^2$$
$$= (1+\epsilon)^2|z_{t^2}-\xi|^2 \left|\int_0^1 \sum_j f_j^{2\epsilon}(z_{t^2}(u))(f_j^2)'(z_{t^2}(u))du\right|^2$$
$$\leq C_{1+\epsilon}d^2(e^{it^2})\left(\int_0^1 \sum_j |z_{t^2}(u)-\xi|^{\frac{\varepsilon_{1+\epsilon}}{2}}|(f_j^2)'(z_{t^2}(u))|du\right)^2$$
$$\leq C_{1+\epsilon}d^{\varepsilon_{1+\epsilon}}(e^{it^2})\left(\int_0^1 \frac{1}{(1-u)^{1-\frac{\varepsilon_{1+\epsilon}}{2}}}du\right)^2 \sum_j \|(f_j^2)'\|^2_{L^2(\Delta_\gamma)}$$
$$\leq C_{1+\epsilon}d^{\varepsilon_{1+\epsilon}}(e^{it^2})\sum_j \|(f_j^2)'\|^2_{L^2(\Delta_\gamma)}.$$

Hence

$$\int_\gamma \sum_j \frac{|f_j^2(z_{t^2})|^{2(1+\epsilon)}}{d(e^{it^2})}dt^2 \leq \sum_j C_\rho \|(f_j^2)'\|^2_{L^2(\Delta_\gamma)}. \quad \text{...(8)}$$

Therefore the result follows from.[6,7 and 8]

In the sequel, we denote by f_j^2 a series of square outer functions in $\mathcal{A}_{\alpha_j^2}$ such that

$$\sum_j \|f_j^2\|_{\mathcal{A}_{\alpha_j^2}} \leq 1$$

and we fix a constant $1+\epsilon$, $0<\epsilon\leq 1$. By [9, Theorem B], we have

$f_j^{2(1+\epsilon)}(f_j)_\Gamma^{2(1+\epsilon)} \in \mathrm{lip}_{\alpha_j^2}$ and

$$\sum_j \left\|f_j^{2(1+\epsilon)}(f_j)_\Gamma^{2(1+\epsilon)}\right\|_{\mathrm{lip}_{\alpha_j^2}} \leq C_{1+\epsilon,1+\epsilon}.$$

To prove Theorem (2.1) we need to estimate the integral

$$\int_{\mathbb{D}} \sum_j \left|f_j^{2(1+\epsilon)}(f_j^{2(1+\epsilon)})'\right|^2 dA(z).$$

Define

$$\sum_j (f_j^2)_\Gamma(z) := \frac{1}{\pi}\int_\Gamma \sum_j \frac{e^{i\theta^2}}{(e^{i\theta^2}-z)^2} \log|f_j^2(e^{i\theta^2})|\, d\theta^2. \quad \text{...(9)}$$

Clearly we have

$$\sum_j (f_j^2)_\Gamma(z) := \frac{1}{\pi}\int_\Gamma \sum_j \frac{e^{i\theta^2}}{(e^{i\theta^2}-z)^2} \log|f_j^2(e^{i\theta^2})|\, d\theta^2. \text{ and}$$

$$\sum_j \left((f_j)_\Gamma^{2(1+\epsilon)}\right)' = \sum_j (1+\epsilon)(f_j)_\Gamma^{2(1+\epsilon)}(g_j^2)_\Gamma,$$

$$\sum_j f_j^{2(1+\epsilon)}(f_j^{2(1+\epsilon)})' = \sum_j (1+\epsilon) f_j^{2(1+\epsilon)}(f_j)_\Gamma^{2(1+\epsilon)}(g_j^2)_\Gamma \quad \text{....(10)}$$

$$= \sum_j f_j^{2\epsilon}(1+\epsilon)(f_j^2)'(f_j)_\Gamma^{(1+\epsilon)} - \sum_j (1+\epsilon) f_j^{2(1+\epsilon)}(f_j)_\Gamma^{2(1+\epsilon)}(g_j^2)_{\mathbb{T}\setminus\Gamma}. \quad \text{...(11)}$$

Since $\sum_j \|f_j^2\|_\infty \leq 1$, it is obvious that

$$\sum_j \left\|(f_j)_\Gamma^{2(1+\epsilon)}\right\|_\infty \leq 1 \text{ and } \sum_j \|f_j^{2\epsilon}\|_\infty \leq 1.$$

Hence, by[11] we get

$$\int_{\mathbb{D}} \sum_j \left|\left(f_j^{2(1+\epsilon)}(f_j)_\Gamma^{2(1+\epsilon)}\right)'\right|^2 dA(z)$$
$$\leq 2(1+\epsilon)^2 \int_{\mathbb{D}} \sum_j \left|\left(f_j^{2(1+\epsilon)}(f_j)_\Gamma^{2(1+\epsilon)}\right)'\right|^2 dA(z).$$

We fix $\gamma = (a, a+\epsilon) \subset T\setminus E_{f_j^2}$

such that $\sum_j f_j^2(a) = \sum_j f_j^2(a+\epsilon) = 0$

Our purpose in what follows is to estimate the integral

$$\int_{\Delta_\gamma} \sum_j \left|\left(f_j^{2(1+\epsilon)}(f_j)_\Gamma^{2(1+\epsilon)}\right)'\right|^2 dA(z) \quad \text{...(13)}$$

which we can rewrite as

$$\int_{\Delta_\gamma} \sum_j \left|\left(f_j^{2(1+\epsilon)}(f_j)_\Gamma^{2(1+\epsilon)}\right)'\right|^2 dA(z) = \int_{\Delta_\gamma^1} + \int_{\Delta_\gamma^2},$$

Where

$$\Delta_\gamma^1 := \{z \in \Delta_\gamma : d(z) < 2(1-|z|)\}$$
$$\Delta_\gamma^2 := \{z \in \Delta_\gamma : d(z) \geq 2(1-|z|)\}.$$

The integral on the region Δ_γ^1. We begin with the following lemma (see[15]).

Lemma (4.2)

$$\int_{\Delta_\gamma} \sum_j \frac{|f_j^2(z) - f_j^2(z/|z|)|^{2(1+\epsilon)}}{(1-|z|)^2} dA(z) \leq \sum_j \frac{1}{2\alpha_j^2\epsilon}\|(f_j^2)'\|_{L^2(\Delta_\gamma)}.$$

Proof

Let z= $(1-\epsilon)e^{it^2} \in \Delta_\gamma$

and put $\varepsilon_{1+\epsilon} = 2\alpha_j^2\epsilon$. We have

$$\sum_j (1-\epsilon)\left|f_j^2\left((1-\epsilon)e^{it^2}\right)-f_j^2\left(e^{it^2}\right)\right|^{2(1+\epsilon)}$$
$$=\sum_j (1-\epsilon)\left|f_j^2\left((1-\epsilon)e^{it^2}\right)-f_j^2\left(e^{it^2}\right)\right|^{2\epsilon}\left|f_j^2\left((1-\epsilon)e^{it^2}\right)-f_j^2\left(e^{it^2}\right)\right|^2$$
$$\leq (1-\epsilon)\epsilon^{1+\varepsilon_{(1+\epsilon)}}\int_{(1-\epsilon)}^{1}\sum_j\left|(f_j^2)'((\tfrac{1}{2}+\epsilon)e^{it^2})\right|^2 d(\tfrac{1}{2}+\epsilon) \leq (1-\epsilon)\epsilon^{1+\varepsilon_{(1+\epsilon)}}\int_{(1-\epsilon)}^{1}\sum_j\left|(f_j^2)'((\tfrac{1}{2}+\epsilon)e^{it^2})\right|^2(\tfrac{1}{2}+\epsilon)\,d(\tfrac{1}{2}+\epsilon)\,.$$

Therefore

$$\int_{\Delta_\gamma}\sum_j\frac{\left|f_j^2(z)-f_j^2(z/|z|)\right|^{2(1+\epsilon)}}{(1-|z|)^2}dA(z) =$$
$$\int_0^1\left(\int_\gamma\sum_j\left|f_j^2\left((1-\epsilon)e^{it^2}\right)-f_j^2\left(e^{it^2}\right)\right|^{2(1+\epsilon)}\frac{(1-\epsilon)dt}{\pi}\right)\frac{d(1-\epsilon)}{\epsilon^2}\leq$$
$$\sum_j\|(f_j^2)'\|_{L^2(\Delta_\gamma)}\int_0^1\frac{1}{\epsilon^{1-\varepsilon_{(1+\epsilon)}}}d(1-\epsilon)\,.$$

This completes the proof.
Now, we can state the following result (see[15]).

Lemma (4.3)

$$\int_{\Delta_\gamma^1}\sum_j\left|f_j^2(z)\right|^{2(1+\epsilon)}\left|\left((f_j^2)_\Gamma\right)'(z)\right|^2 dA(z)\leq\sum_j C_{(1+\epsilon)}\|(f_j^2)'\|_{L^2(\Delta_\gamma)}^2.$$

Proof
By Cauchy's estimate, it follows that

$$\sum_j|((f_j^2)_\Gamma)'((1-\epsilon)e^{it^2})|\leq\frac{1}{\epsilon}$$

Using Lemma (4.2), we get

$$\int_{\Delta_\gamma^1}\sum_j\left|f_j^2(z)\right|^{2(1+\epsilon)}\left|((f_j^2)_\Gamma)'(z)\right|^2 dA(z)\leq\int_{\Delta_\gamma^1}\sum_j\frac{\left|f_j^2(z)\right|^{2(1+\epsilon)}}{(1-|z|)^2}dA(z)$$
$$\leq\sum_j C_{(1+\epsilon)}\|(f_j^2)'\|_{L^2(\Delta_\gamma)}^2$$
$$+2^{(2\epsilon+1)}\int_{\Delta_\gamma^1}\sum_j\frac{\left|f_j^2(z/|z|)\right|^{2(1+\epsilon)}}{(1-|z|)^2}dA(z).\qquad\ldots(14)$$

Using Lemma (4.1), we obtain

$$\int_{\Delta_\gamma^1}\sum_j\frac{\left|f_j^2(z/|z|)\right|^{2(1+\epsilon)}}{(1-|z|)^2}dA(z)=\frac{1}{\mu}\int_{\Delta_\gamma^1}\sum_j\frac{\left|f_j^2\left(e^{it^2}\right)\right|^{2(1+\epsilon)}}{\epsilon^2}(1-\epsilon)d(1-\epsilon)dt^2$$
$$\leq\frac{C}{\pi}\int_\gamma\sum_j\frac{\left|f_j^2\left(e^{it^2}\right)\right|^{2(1+\epsilon)}}{\epsilon^2}dt^2$$
$$\leq\sum_j C_{(1+\epsilon)}\|(f_j^2)'\|_{L^2(\Delta_\gamma)}^2.\qquad\ldots(15)$$

The result of our lemma follows by combining the estimates.[14 and 15]

The integral on the region Δ_γ^2 . In this subsection, we estimate the integral

$$\int_{\Delta_\gamma^2}\sum_j\left|f_j^2(z)\right|^{2(1+\epsilon)}\left|((f_j^2)_\Gamma)'(z)\right|^2 dA(z)$$

Before this, we make some remarks. For $z\in D$ define

$$a_\gamma(z):=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_\Gamma\sum_j\frac{-\log\left|f_j^2\left(e^{it^2}\right)\right|}{\left|e^{i\theta^2}-z\right|^2}d\theta^2 & if\ \gamma\not\subseteq\Gamma\\[2ex] \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{T}\setminus\Gamma}\sum_j\frac{-\log\left|f_j^2\left(e^{it^2}\right)\right|}{\left|e^{i\theta^2}-z\right|^2}d\theta^2 & if\ \gamma\not\subseteq\Gamma.\end{cases}$$

Using the equation,[10] it is easy to see that

$$\sum_j\left|f_j^2(z)^{1+\epsilon}((f_j^2)_\Gamma)'(z)\right|^2$$
$$\leq 4\sum_j\left|f_j^2(z)^{1+\epsilon}\frac{1}{2\pi}\int_\Gamma\frac{-\log\left|f_j^2\left(e^{it^2}\right)\right|}{\left|e^{i\theta^2}-z\right|^2}d\theta^2\right|^2\qquad\ldots(16)$$

Using the equation,[11] it is clear that

$$\sum_j\left|f_j^2(z)^{1+\epsilon}((f_j^2)_\Gamma)'(z)\right|^2$$
$$\leq 2\sum_j\left|(f_j^2)'(z)\right|^2+8\sum_j\left|f_j^2(z)^{1+\epsilon}\frac{1}{2\pi}\int_{\mathbb{T}\setminus\Gamma}\frac{-\log\left|f_j^2\left(e^{it^2}\right)\right|}{\left|e^{i\theta^2}-z\right|^2}d\theta^2\right|^2\qquad\ldots(17)$$

Then

$$\int_{\Delta_\gamma^2}\sum_j\left|f_j^2(z)\right|^{2(1+\epsilon)}\left|((f_j^2)_\Gamma)'(z)\right|^2 dA(z)$$
$$\leq 2\sum_j\|(f_j^2)'\|_{L^2(\Delta_\gamma)}^2+8\int_{\Delta_\gamma^2}\sum_j f_j^2(z)^{2(1+\epsilon)}a_\gamma^2(z)dA(z).\qquad\ldots(18)$$

Since $\log|f_j^2|\in L^1(\mathbb{T})$, we have

$$a_\gamma(z)\leq\frac{C}{d^2(z)}\qquad(z\in\Delta_\gamma)\qquad\ldots(19)$$

Given such inequality, it is not easy to estimate immediately the integral of the series of functions

$\sum_j|f_j^2(z)|^{2(1+\epsilon)}a_\gamma^2(z)$ on the whole Δ_γ^2

In what follows, we give a partition of Δ_γ^2 into three parts so that one can estimate the integral

$\int \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} a_\gamma^2(z)dA(z)$ on each part. Let $\epsilon\ \Delta_\gamma^2$, three situations are possible :

$$a_\gamma(z) \le 8^{\frac{|\log(d(z))|}{d(z)}} \quad ...(20)$$

$$8^{\frac{|\log(d(z))|}{d(z)}} < a_\gamma(z) < 8^{\frac{|\log(d(z))|}{\epsilon}} \quad ...(21)$$

$$8^{\frac{|\log(d(z))|}{\epsilon}} \le a_\gamma(z) \quad ...(22)$$

We can now Δ_γ^2 into the following three parts

$$\Delta_\gamma^{21} := \{z \in \Delta_\gamma^2 : z \text{ satisfying } (20)\},$$
$$\Delta_\gamma^{22} := \{z \in \Delta_\gamma^2 : z \text{ satisfying } (21)\},$$
$$\Delta_\gamma^{23} := \{z \in \Delta_\gamma^2 : z \text{ satisfying } (22)\},$$

The integral on the regions Δ_γ^{21} and Δ_γ^{23} . In this case we begin by the following (see[15])

Lemma (4.4):

$$\int_{\Delta_\gamma^{21}} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} a_\gamma^2(z)dA(z) \le \sum_j C_{(1+\epsilon)} \left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2$$

Proof

Using Lemma (4.2), we get

$$\int_{\Delta_\gamma^{21}} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} a_\gamma^2(z)dA(z)$$
$$\le 2^{(1+\epsilon)} \int_{\Delta_\gamma^{21}} \sum_j \left|f_j^2(z)\right|^{\epsilon} \left|f_j^2(z) - f_j^2(z/|z|)\right|^{(\epsilon+2)} a_\gamma^2(z)dA(z)$$
$$+ 2^{(1+\epsilon)} \int_{\Delta_\gamma^{21}} \sum_j \left|f_j^2(z)\right|^{j} \left|f_j^2(z/|z|)\right|^{\epsilon+2} a_\gamma^2(z)dA(z)$$
$$\le C_{1+\epsilon} \int_{\Delta_\gamma} \sum_j \frac{\left|f_j^2(z) - f_j^2(z/|z|)\right|^{\epsilon+2}}{(1-|z|)^2} dA(z)$$
$$+ C_{1+\epsilon} \int_{\Delta_\gamma^{21}} \sum_j \frac{\left|f_j^2(e^{it^2})\right|^{\epsilon+2}}{d^2(e^{it^2})} (1-\epsilon)d(1-\epsilon)dt^2$$
$$\le \sum_j C_{1+\epsilon} \left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2 + C_{1+\epsilon} \int_{\Delta_\gamma^{21}} \sum_j \frac{\left|f_j^2(e^{it^2})\right|^{\epsilon+2}}{d^2(e^{it^2})} d(1-\epsilon)dt^2 = I_{2,1}$$

Let $e^{it^2} \in \gamma$ and denote by $(z-2\epsilon)_{t^2}$ the point of

$\partial\Delta_\gamma^2 \cap \mathbb{D}$ such that

$(z-2\epsilon)_{t^2}/|(z-2\epsilon)_{t^2}| = e^{it^2}$ We have

$$\left|e^{it^2} - (z-2\epsilon)_{t^2}\right| = 1 - |(z-2\epsilon)_{t^2}| = \frac{d((z-2\epsilon)_{t^2})}{2} \le d(e^{it^2}).$$

Then

$$\int_{\Delta_\gamma^{21}} \sum_j \frac{\left|f_j^2(e^{it^2})\right|^{\epsilon+2}}{d^2(e^{it^2})} d(1-\epsilon)dt^2 \le \int_{\Delta_\gamma^2} \sum_j \frac{\left|f_j^2(e^{it^2})\right|^{\epsilon+2}}{d^2(e^{it^2})} d(1-\epsilon)dt^2$$
$$= \int_\gamma \sum_j \frac{\left|f_j^2(e^{it^2})\right|^{\epsilon+2}}{d^2(e^{it^2})} \int_{|(z-2\epsilon)_{t^2}|}^1 d(1-\epsilon)dt^2 \le \int_\gamma \sum_j \frac{\left|f_j^2(e^{it^2})\right|^{\epsilon+2}}{d^2(e^{it^2})} dt^2$$

Using Lemma (4.1), we get

$I_{2,1} \le \sum_j C_{1+\epsilon} \left\|(f_j^2)'\right\|_{L^2(\Delta_\gamma)}^2$. This proves the result.

Lemma (4.5)

$$\int_{\Delta_\gamma^{23}} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} a_\gamma^2(z)dA(z) \le CA(\Delta_\gamma),$$

where A $(\Delta_\gamma)$is the area measure of Δ_γ .

Proof

Set

$$\Lambda_\gamma := \begin{cases} \Gamma & \text{for } \gamma \not\subseteq \Gamma, \\ \mathbb{T}\setminus\Gamma & \text{for } \gamma \subseteq \Gamma. \end{cases}$$

Let $\epsilon\ \Delta_\gamma^{23}$ We have

$$\sum_j \left|f_j^2(z)\right| = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \sum_j \frac{2\epsilon-\epsilon^2}{\left|e^{i\theta^2}-z\right|^2} \log\left|f_j^2(e^{i\theta^2})\right| d\theta^2\right\}$$
$$\le \exp\left\{\frac{1}{2\pi}\int_{\Lambda_\gamma} \sum_j \frac{2\epsilon-\epsilon^2}{\left|e^{i\theta^2}-z\right|^2} \log\left|f_j^2(e^{i\theta^2})\right| d\theta^2\right\} = \exp\{-\epsilon a_\gamma(z)\} \le d^8(z).$$

Using,[19] we obtain the result.

The integral on the region $\epsilon\ \Delta_\gamma^{23}$. Here, we will give an estimate of the following integral

$$\int_{\Delta_\gamma^{22}} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} a_\gamma^2(z)dA(z).$$

Before doing this, we begin with some lemmas (see[15]).

The next one is essential for what follows. Note that a similar result is used by different authors: Korenblum,[8] Matheson,[9] Shamoyan,[11] and Shirokov.[13, 12]

Lemma (4.6)

Let $z \in \Delta_\gamma^{22}$ and let μ_z= $1 - \frac{8|\log(d(z))|}{a_\gamma(z)}$. Then

$$\sum_j \left|f_j^2(\mu_z z)\right| \le d^2(z) \quad ...(23)$$

Proof

Let $z \in \Delta_\gamma$ and let $\mu<1$. We have

$$\sum_j \left|f_j^2(\mu_z)\right| = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\sum_j \frac{1-(\mu(1-\epsilon))^2}{\left|e^{i\theta^2}-\mu z\right|^2}\log\left|f_j^2\left(e^{i\theta^2}\right)\right|d\theta^2\right\}$$
$$\le \exp\left\{\frac{1}{2\pi}\int_{\Lambda_\gamma}\sum_j \frac{1-(\mu(1-\epsilon))^2}{\left|e^{i\theta^2}-\mu z\right|^2}\log\left|f_j^2\left(e^{i\theta^2}\right)\right|d\theta^2\right\}$$
$$= \exp\left\{-(1-\mu(1-\epsilon))\inf_{\theta^2\in\Lambda_\gamma}\left|\frac{e^{i\theta^2}-z}{e^{i\theta^2}-\mu z}\right|^2 a_\gamma(z)\right\}.$$

For $z \in \Delta_\gamma^{22}$ it is clear that $1-\mu z \le d(z) \le |e^{i\theta^2}-z|$ for all

$e^{i\theta^2} \in \Lambda_\gamma$.

Then

$$\inf_{\theta^2\in\Lambda_\gamma}\left|\frac{e^{i\theta^2}-z}{e^{i\theta^2}-\mu z}\right|^2 \ge \frac{1}{2} \qquad (z \in \Delta_\gamma^{22}).$$

Thus

$$\sum_j \left|f_j^2(\mu_z z)\right| \le \exp\left\{-\frac{1-\mu_z}{4}a_\gamma(z)\right\} \qquad (z \in \Delta_\gamma^{22}).$$

Then, we have

$$\sum_j \left|f_j^2(\mu_z z)\right| \le \exp\left\{-\tfrac{1}{4}(1-\mu_z)a_\gamma(z)\right\} = d^2(z) \qquad (z \in \Delta_\gamma^{22}),$$

which yields.[23]

For $\epsilon > 0$ define $\gamma_{(1-\epsilon)} := \{z \in \mathbb{D}: |z| = 1-\epsilon \text{ and } z/|z| \in \gamma\}$

Without loss of generality, we can suppose that

$d(z) \le \frac{1}{2}$, $z \in \Delta_\gamma^2$. We need the following (see[15]).

Note that: we deduce that

$$\sum_j \left|f_j^2(\mu_z z)\right| \le \frac{c'}{\left\|\log\left(\frac{1}{2}\right)\right\|} \quad \text{where } c' = \frac{c}{16}.$$

Lemma (4.7)

Let $\epsilon > 0$. Then

$$\int_{\Delta_\gamma^{22}}\sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} a_\gamma^2(z)\,dA(z) \le \sum_j C_{1+\epsilon}\left\|(f_j^2)'\right\|^2_{L^2(\Delta_\gamma)} + CA(\Delta_\gamma).$$

Proof

Let $(1-\epsilon)e^{it^2} \in \Delta_\gamma^{22}$. Then

$$\sum_j \left|f_j^2\left((1-\epsilon)e^{it^2}\right) - f_j^2\left(\mu_{(1-\epsilon)e^{it^2}}(1-\epsilon)e^{it^2}\right)\right|^{\epsilon}\left[\left(1-\mu_{(1-\epsilon)e^{it^2}}\right)a_\gamma\left((1-\epsilon)e^{it^2}\right)\right]^2$$
$$\le 64\left(1-\mu_{(1-\epsilon)e^{it^2}}\right)^{\varepsilon(1+\epsilon)}\log^2\left(d\left((1-\epsilon)e^{it^2}\right)\right) \le C_{1+\epsilon}.$$

It is clear that

$\epsilon \le 1-\mu_{(1-\epsilon)e^{it^2}} \le d\left((1-\epsilon)e^{it^2}\right) \le \frac{1}{2}$ and so

$\frac{1}{2} \le d\left((1-\epsilon)e^{it^2}\right) \le (1-\epsilon)$.

We have

$$\int_{\gamma_{(1-\epsilon)}\cap\Delta_\gamma^{22}}\sum_j \left|f_j^2\left((1-\epsilon)e^{it^2}\right) - f_j^2\left(\mu_{(1-\epsilon)e^{it^2}}(1-\epsilon)e^{it^2}\right)\right|^{2(1+\epsilon)} a_\gamma^2\left((1-\epsilon)e^{it^2}\right)(1-\epsilon)dt^2$$
$$\le C_{1+\epsilon}\int_{\gamma_{(1-\epsilon)}\cap\Delta_\gamma^{22}}\sum_j \frac{\left|f_j^2\left((1-\epsilon)e^{it^2}\right) - f_j^2\left(\mu_{(1-\epsilon)e^{it^2}}(1-\epsilon)e^{it^2}\right)\right|^{\epsilon+2}}{\left(1-\mu_{(1-\epsilon)e^{it^2}}\right)^2}(1-\epsilon)dt^2$$
$$\le \frac{C_{1+\epsilon}}{\epsilon^{1-\varepsilon(1+\epsilon)}}\int_{\gamma_{(1-\epsilon)}\cap\Delta_\gamma^{22}}\left(\int_{\mu_{(1-\epsilon)e^{it^2}}(1-\epsilon)}^{(1-\epsilon)}\sum_j\left|(f_j^2)'\left(\left(\tfrac{1}{2}+\epsilon\right)e^{it^2}\right)\right|^2 d\left(\tfrac{1}{2}+\epsilon\right)\right)(1-\epsilon)dt^2 \le \frac{C_{1+\epsilon}}{\epsilon^{1-\varepsilon(1+\epsilon)}}\int_{\left(\frac{1}{2}+\epsilon\right)_{(1-\epsilon)}}\sum_j\left|(f_j^2)'\left(\left(\tfrac{1}{2}+\epsilon\right)e^{it^2}\right)\right|^2\left(\tfrac{1}{2}+\epsilon\right)d\left(\tfrac{1}{2}+\epsilon\right)dt^2$$
$$\le \frac{C_{1+\epsilon}}{\epsilon^{1-\varepsilon(1+\epsilon)}}\int_{\left(\frac{1}{2}+\epsilon\right)_{(1-\epsilon)}}\sum_j\left|(f_j^2)'(z-\epsilon)\right|^2 dA(z-\epsilon),$$

Where

$$S_{(1-\epsilon)} := \left\{(z-\epsilon) \in \mathbb{D} : 0 \le |z-\epsilon| \le (1-\epsilon) \text{ and } \frac{z-\epsilon}{|z-\epsilon|} \in \gamma\right\}.$$

The proof is therefore completed.

The last result that we need before giving the proof of Theorem (2.1) is the following one (see[15]).

Lemma (4.8)

$$\sum_j \left|f_j^2\left((1-\epsilon)e^{it^2}\right) - f_j^2\left(\mu_{(1-\epsilon)e^{it^2}}(1-\epsilon)e^{it^2}\right)\right|^{\epsilon}\left[\left(1-\mu_{(1-\epsilon)e^{it^2}}\right)a_\gamma\left((1-\epsilon)e^{it^2}\right)\right]^2$$
$$\le 64\left(1-\mu_{(1-\epsilon)e^{it^2}}\right)^{\varepsilon(1+\epsilon)}\log^2\left(d\left((1-\epsilon)e^{it^2}\right)\right) \le C_{1+\epsilon}.$$

Proof

Using [19] and Lemmas (4.6) and (4.7), we find that

$$\int_{\Delta_\gamma^{22}}\sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} a_\gamma^2(z)\,dA(z)$$
$$= \frac{1}{\pi}\int_0^1\left(\int_{\gamma_{(1-\epsilon)}\cap\Delta_\gamma^{22}}\sum_j\left|f_j^2\left((1-\epsilon)e^{it^2}\right)\right|^{2(1+\epsilon)} a_\gamma^2\left((1-\epsilon)e^{it^2}\right)(1-\epsilon)dt^2\right)d(1-\epsilon)$$
$$\le CA(\Delta_\gamma)$$
$$+\,2^{(2\epsilon+1)}\int_0^1\left(\int_{\gamma_{(1-\epsilon)}\cap\Delta_\gamma^{22}}\sum_j\left|f_j^2\left((1-\epsilon)e^{it^2}\right) - f_j^2\left(\mu_{(1-\epsilon)e^{it^2}}(1-\epsilon)e^{it^2}\right)\right|^{2(1+\epsilon)} a_\gamma^2\left((1-\epsilon)e^{it^2}\right)(1-\epsilon)dt^2\right)d(1-\epsilon)$$
$$\le CA(\Delta_\gamma) + \sum_j C_{1+\epsilon}\left\|(f_j^2)'\right\|^2_{L^2(\Delta_\gamma)}.$$

This completes the proof of the lemma.

Conclusion

Now, according to (18) and Lemmas (4.4), (4.5) and (4.8), we obtain

$$\begin{aligned}\int_{\gamma_{(1-\epsilon)}\cap\Delta_\gamma^{22}} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} \left|((f_j^2)_\Gamma)'(z)\right|^2 dA(z) &\le 2\sum_j \|(f_j^2)'\|^2_{L^2(\Delta_\gamma)} + 8\int_{\gamma_{(1-\epsilon)}\cap\Delta_\gamma^{22}} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} a_\gamma^2(z)\,dA(z)\\ &\le \sum_j C_{1+\epsilon}\|(f_j^2)'\|^2_{L^2(\Delta_\gamma)} + CA(\Delta_\gamma).\end{aligned}$$

Combining this with Lemma (4.3), we deduce that

$$\int_{\Delta_\gamma} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} \left|((f_j^2)_\Gamma)'(z)\right|^2 dA(z) \le \sum_j C_{1+\epsilon}\|(f_j^2)'\|^2_{L^2(\Delta_\gamma)} + CA(\Delta_\gamma).$$

Hence

$$\begin{aligned}&\int_{\mathbb{D}} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} \left|((f_j^2)_\Gamma)'(z)\right|^2 dA(z)\\ &= \sum_{n=1}^{\infty}\int_{\Delta_{\gamma_n}} \sum_j \left|f_j^2(z)\right|^{2(1+\epsilon)} \left|((f_j^2)_\Gamma)'(z)\right|^2 dA(z)\\ &\le \sum_j C_{1+\epsilon}\sum_{n=1}^{\infty}\|(f_j^2)'\|^2_{L^2(\Delta_{\gamma_n})} + C\sum_{n=1}^{\infty} A(\Delta_{\gamma_n}) \le C_{1+\epsilon}.\end{aligned}$$

This completes the proof of Theorem (2.1)

Acknowledgement

The author wishes to thank all the participants who were involved in this study.

Conflict of Interest

The authors do not have any conflict of interest.

References

1. B.Bouya, Id´eaux ferm´es de certaines alg`ebres de fonctions analytiques, C. R. Math. *Acad. Sci.* Paris 343 (2006), no. 4, 235–238.
2. L.Carleson, A representation formula in the Dirichlet space, *Math*. Z. 73 (1960), 190–196.
3. P.L.Duren, Theory of Hp spaces, Academic Press, New York, 1970.
4. O.El-Fallah, K. Kellay, T. Ransford, Cyclicity in the Dirichlet space, *Ark. Mat*. 44 (1) (2006), 61–86.
5. J. Esterle, E. Strouse, F. Zouakia, Closed ideal of A+ and the Cantor set, *J. reine angew. Math*. 449 (1994), 65–79.
6. H. Hedenmalm, A. Shields, Invariant subspaces in Banach spaces of ana- lytic functions, Mich. *Math*. J. 37 (1990), 91–104.
7. K.Hoffman, Banach spaces of analytic functions, Dover Publications Inc., New York, 1988, Reprint of the 1962 original.
8. B. I. Korenblum, Invariant subspaces of the shift operator in a weighted Hilbert space, *Mat. Sb.* 89(131)(1972), 110–138.
9. A. Matheson, Approximation of analytic functions satisfying a Lipschitz condition, *Mich. Math. J.* 25 (1978), no. 3, 289–298.
10. W.Rudin, Real and complex analysis, second ed., McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York, 1974.
11. F. A. Shamoyan, Closed ideals in algebras of functions that are analytic in the disk and smooth up to its boundary, Mat. Sb. 79 (1994), no. 2, 425–445. Closed Ideals In Some Algebras of Analytic Functions 19
12. N.A.Shirokov, Analytic functions smooth up to the boundary, Lecture Notes in Mathematics, 1312. Springer-Verlag, Berlin, 1988.
13. N. A. Shirokov, Closed ideals of algebras of B_ pq–type, (Russian) Izv. Akad. Nauk. SSSR, *Mat.* 46 (1982), no. 6, 1316–1333.
14. B. A. Taylor, D.L. Williams, Ideals in rings of analytic functions with smooth boundary values, *Can. J. Math*. 22 (1970), 1266–1283. E-mail address: brahimbouya@gmail.com
15. Brahim Bouya ,Closed ideals in some algebras of analytic function, arXiv:0802.0890v1 [math.CV] 6 Feb 2008.