



Hamilton-Jacobi Equation of Time Dependent Hamiltonians

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Abstract

In this work, we apply the geometric Hamilton-Jacobi theory to obtain solution of Hamiltonian systems in classical mechanics that are either compatible with two structures: the first structure plays a central role in the theory of time- dependent Hamiltonians, whilst the second is used to treat classical Hamiltonians including dissipation terms. It is proved that the generalization of problems from the calculus of variation methods in the non stationary case can be obtained naturally in Hamilton-Jacobi formalism.



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Introduction

Since Bateman proposed the time-dependent Hamiltonian in a classical context¹ for the illustration of dissipative systems, there has been much attention paid to quantum-mechanical treatments of nonlinear and non conservative systems. In studying nonlinear systems, it is essential to introduce a time-dependent Hamiltonian which describes the frictional cases. This was discovered first by Caldirola,² and rederived independently by Kanai³ via Bateman's dual Hamiltonian, and afterward by several others.⁴

Hamilton Jacobi equations (HJE) are nonlinear first order equations which have been first introduced in classical mechanics, but find applications in many other fields of mathematics. Our interest in these equations lies mainly in the connection with calculus of variations and optimal control.

However, Hamilton-Jacobi method has been studied for a wide range of systems with time-independent

Hamiltonians. For systems with time-dependent Hamiltonians, however, due to the complexity of dynamics, little has been known about quantum of action variables.

However, Hamilton-Jacobi theory builds a bridge between classical mechanics and other branches of physics. Mainly, the Hamilton-Jacobi equation can be viewed as a precursor to the Schrödinger equation.⁵⁻¹¹

Our primary goals will be to extend the HJ formulation for time-dependent systems, building on the previous work by Rabei *et al.* (2002), the idea is to construct the Hamiltonian function and the corresponding equation of motion for dissipative systems. The methodology for that, the principal function is determined using the method of separation of variables. The equation of motion can then be readily obtained.

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Hamilton-Jacobi Formalism

We start with the Lagrangian

$$L = L_0(q, \dot{q}) e^{\lambda t} \quad \dots(1)$$

Here $L_0(q, \dot{q})$ stands for the usual Lagrangian and λ is the dissipation factor. The generalized momentum is defined by.¹²

$$p_i = \partial L / \partial \dot{q}_i \quad \dots(2)$$

The corresponding Hamiltonian is

$$H(q, p, t) = p \dot{q} - L \quad \dots(3)$$

Hamilton's Jacobi equation is differential equation of the form:

$$H(q_1, \dots, q_n; \partial S / \partial q_1, \dots, \partial S / \partial q_n; t) + \partial S / \partial t = 0 \quad \dots(4)$$

It is a partial differential of (n+1) variables, $q_1, \dots, q_n; t$.

The complete solution of Eq. (4) can be written in the form⁶

$$S = S(q_1, \dots, q_n; \alpha_1, \dots, \alpha_n; t) \quad \dots(5)$$

Eq. (5) presents S as a function of n coordinates, the time t, and n independent quantities α_i .

We can take the n constants of integration to be constants of momenta:

$$p_i = \alpha_i \quad \dots(6)$$

$$p_i = \partial S(q_i, \alpha_i, t) / (\partial \dot{q}_i) \quad \dots(7)$$

The relationship between p and q then describes the orbit in phase space in terms of these constants of motion, furthermore the quantities

$$\partial S / (\partial q_i) = p_i \quad \dots(8)$$

Are the equations also constants of motion, and these equations can be inverted to find q as a function of all α and β constants and time.

Thus, the Hamilton-Jacobi function is given by

$$H(q, p) + \partial S(q, t) / (\partial t) = 0 \quad \dots(10)$$

The resulting action S is

$$S = \int L dt + \text{constant} \quad \dots(11)$$

or

$$S = \int L_0 e^{\lambda t} dt = \int (p_i \dot{q}_i - H) dt \quad \dots(12)$$

We must write S in the separable form

$$S(q, \alpha, t) = W(q, \alpha) + f(t) \quad \dots(13)$$

The time-independent function W(q, α) is

sometimes called Hamilton characteristic function.

Differentiating Eq. (13) with respect to t, we find that

$$\partial S / \partial t = \partial f / \partial t \quad \dots(14)$$

From Eq. (10), it follows that

$$\partial f / \partial t = -H \quad \dots(15)$$

Therefore, the time derivative $\partial S / \partial t$ in HJE must be a constant, usually denoted by $(-\alpha)$.

$$S(q, \alpha, t) = W(q, \alpha) - \alpha(t) \quad \dots(16)$$

It follows that

$$H(q_i, \partial W / (\partial q_i)) = \alpha_i \quad \dots(17)$$

The equations of transformation are

$$p_i = \partial W / (\partial q_i) \quad \dots(18)$$

$$Q_i = \partial W / (\partial \alpha_i) \quad \dots(19)$$

While these equations resemble Eq. (7) and (8) respectively for Hamilton's principal function S, the condition now determining W is that it is the new canonical momentum α_i

$$H(q_i, p_i) = \alpha_i \quad \dots(20)$$

Examples

Friction Linear in the Velocity

The Lagrangian depending on time is.¹³⁻¹⁴

$$L = L = 1/2 e^{\lambda t} x^2 \quad \dots(21)$$

The linear momentum is given by

$$p = \frac{\partial L}{\partial \dot{x}} = e^{\lambda t} \dot{x} \quad \dots(22)$$

This equation can readily be solving to give

$$\dot{x} = p e^{-\lambda t} \quad \dots(23)$$

The canonical Hamiltonian has the standard form

$$H = p\dot{x} - L \quad \dots(24)$$

Now, substituting Eqs. (21) and (23) into (24), we get

$$H = \frac{p^2}{2} e^{-\lambda t} \quad \dots(25)$$

The momentum can be computed from

$$p = \partial S / \partial x \quad \dots(26)$$

Substituting Eq. (26) into Eq. (25), we find that

$$H = \frac{1}{2} \left(\frac{\partial S}{\partial x} \right)^2 e^{-\lambda t} \quad \dots(27)$$

The Hamilton-Jacobi equation is

$$H + \partial S / \partial t = 0 \quad \dots(28)$$

Substituting Eq. (27) into Eq. (28), we get

$$\frac{1}{2} \left(\frac{\partial S}{\partial x} \right)^2 e^{-\lambda t} + \frac{\partial S}{\partial t} = 0 \quad \dots(29)$$

With the change of variable,

$$\tau = -\frac{1}{2\lambda} e^{-\lambda t} \quad \dots(30)$$

we can eliminate the factor τ in Eq. (29) which is transformed in

$$\left(\frac{\partial S(x,\tau)}{\partial x} \right)^2 + \frac{\partial S(x,\tau)}{\partial \tau} = 0 \quad \dots(31)$$

Now it is possible to propose

$$S(x,\alpha,\tau) = W(x,\alpha) - \alpha\tau \quad \dots(32)$$

That is

$$\frac{\partial S}{\partial x} = \frac{\partial W}{\partial x} \quad \dots(33)$$

$$\frac{\partial S}{\partial \tau} = -\alpha \quad \dots(34)$$

Substituting Eqs. (33) and (34) into Eq. (31), we get

$$\left(\frac{\partial W}{\partial x} \right)^2 - \alpha = 0 \quad \dots(35)$$

So that

$$\left(\frac{\partial W}{\partial x} \right)^2 = \alpha \quad \dots(36)$$

Taking the square root of each side of Eq. (36), we have

$$\partial W / \partial x = \sqrt{\alpha} \quad \dots(37)$$

Integrating Eq. (37), we have

$$W = \sqrt{\alpha} x \quad \dots(38)$$

Substituting this value of W into Eq. (32), we get

$$S = \sqrt{\alpha} x - \alpha\tau \quad \dots(39)$$

Returning to the variable t results in

$$S = \sqrt{\alpha} x + \alpha \frac{1}{2\lambda} e^{-\lambda t} \quad \dots(40)$$

Then we can obtain

$$p = \partial S / \partial x = \sqrt{\alpha} \quad \dots(41)$$

and

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{x}{2\sqrt{\alpha}} + \frac{1}{2\lambda} e^{-\lambda t} \quad \dots(42)$$

We impose the initial condition $x_0 = 0$ for $t = 0$, which gives

$$\beta_0 = 1 / 2\lambda \quad \dots(43)$$

Substituting Eq. (42) into Eq. (43) we can get the expression for x,

$$\frac{1}{2\lambda} = \frac{x}{2\sqrt{\alpha}} + \frac{1}{2\lambda} e^{-\lambda t} \quad \dots(44)$$

Then

$$x = \frac{\sqrt{\alpha}}{\lambda} (1 - e^{-\lambda t}) \quad \dots(45)$$

The value for α is set from equation (41), taking $p = p_0$ for $t = 0$, that is

$$\sqrt{\alpha} = p_0 = v_0 \quad \dots(46)$$

Finally, we have

$$x(t) = \frac{v_0}{\lambda} (1 - e^{-\lambda t}) \quad \dots(47)$$

In fact, this result is in agreement with that obtained by Euler's equation.

Friction Quadratic in the Velocity

It is also known that the equation of motion for a particle with Newtonian friction $f = -m\lambda v^2$ can be derived from the Lagrangian.^{14,15}

$$L = \frac{\dot{x}^2}{2} e^{2\lambda x} \quad \dots(48)$$

The linear momentum is given by

$$p = \partial L / \partial \dot{x} = \dot{x} e^{2\lambda x} \quad \dots(49)$$

This equation can readily be solved to give

$$\dot{x} = p e^{-2\lambda x} \quad \dots(50)$$

The canonical Hamiltonian has the standard form

$$H = p \dot{x} - L \quad \dots(51)$$

Equation (51) becomes

$$H = \frac{p^2}{2} e^{-2\lambda x} \quad \dots(52)$$

The momentum can be computed from

$$p = \partial S / \partial x \quad \dots(53)$$

Substituting Eq. (53) into Eq. (52)

$$H = \frac{1}{2} \left(\frac{\partial S}{\partial x} \right)^2 e^{-2\lambda x} \quad \dots(54)$$

With this Hamiltonian we can write the Hamilton-Jacobi equation

$$\frac{1}{2} \left(\frac{\partial S(x,\alpha,t)}{\partial x} \right)^2 e^{-2\lambda x} + \frac{\partial S(x,\alpha,t)}{\partial t} = 0 \quad \dots(55)$$

It is possible to propose

$$S(x,\alpha,t) = W(x,\alpha) - \alpha t \quad \dots(56)$$

That is

$$\partial S / \partial x = \partial W / \partial x \quad \dots(57)$$

And

$$\partial S / \partial t = -\alpha \quad \dots(58)$$

Substituting Eqs. (57) and (58) into (55), we get

$$\frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 e^{-2\lambda x} = \alpha \quad \dots(59)$$

so that

$$\left(\frac{\partial W}{\partial x} \right)^2 = 2\alpha e^{2\lambda x} \quad \dots(60)$$

Taking the square root of each side Eq. (60), we have

$$\frac{\partial W}{\partial x} = \sqrt{2\alpha} e^{\lambda x} \quad \dots(61)$$

Integrating Eq. (61), we get

$$W = \frac{\sqrt{2\alpha}}{\lambda} e^{\lambda x} \quad \dots(62)$$

Substituting Eq. (62) into Eq. (56), we have

$$S = \frac{\sqrt{2\alpha}}{\lambda} e^{\lambda x} - \alpha t \quad \dots(63)$$

Applying the usual method, we obtain

$$p = \frac{\partial S}{\partial x} = \sqrt{2\alpha} e^{\lambda x} \quad \dots(64)$$

And

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{1}{\lambda\sqrt{2\alpha}} e^{\lambda x} - t \quad \dots(65)$$

Since $v = p e^{-2\lambda x}$, if we take $x(t=0) = 0$, then $p_0 = v_0$

the Eq.(64) is

$$v_0 = \sqrt{2\alpha} e^{\lambda x} \quad \dots(66)$$

Taking the square of each sides and substituting $x = 0$

$$v_0^2 = 2\alpha \quad \dots(67)$$

$$\alpha = \frac{v_0^2}{2} \quad \dots(68)$$

Substituting Eq. (68) into Eq. (65) when $x(t=0)=0$, we get

$$\beta = \frac{1}{v_0\lambda} \quad \dots(69)$$

Finally, we can obtain x from the Eqs. (65), (68) and (69) $S(x, \alpha, t) = W(x, \alpha) T(t)$... (81)

$$\frac{1}{v_0 \lambda} = \frac{1}{\sqrt{v_0^2 \lambda}} e^{\lambda x - t} \quad \dots(70)$$

$$e^{\lambda x} = 1 + v_0 \lambda t \quad \dots(71)$$

Taking the logarithm of each side

$$x = \frac{1}{\lambda} \ln(1 + v_0 \lambda t) \quad \dots(72)$$

In fact, this result is in agreement with that obtained by Euler's equation

Linearly Damped Particle with Constant Force
A suitable Lagrangian for the linearly damped particle moving in one dimension under a constant force is.¹⁶

$$L = e^{\gamma t} \left(\frac{1}{2} \dot{x}^2 - gx \right) \quad \dots(73)$$

The linear momentum is given by

$$p = \frac{\partial L}{\partial \dot{x}} = e^{\gamma t} \dot{x} \quad \dots(74)$$

This equation can readily be solved to give

$$x = p e^{-\gamma t} \quad \dots(75)$$

The Hamiltonian is

$$H = p \dot{x} - L \quad \dots(76)$$

Substituting Eqs. (73) and (75) into Eq. (76)

$$H = \frac{1}{2} P^2 e^{-\gamma t} + gx e^{\gamma t} \quad \dots(77)$$

The conjugate momentum is then

$$p = \frac{\partial S}{\partial x} \quad \dots(78)$$

Eq. (77) becomes

$$H = \frac{1}{2} \left(\frac{\partial S}{\partial x} \right)^2 e^{-\gamma t} + gx e^{\gamma t} \quad \dots(79)$$

The HJE is

$$\frac{1}{2} \left(\frac{\partial S}{\partial x} \right)^2 e^{-\gamma t} + gx e^{\gamma t} + \frac{\partial S}{\partial t} = 0 \quad \dots(80)$$

Where $S = S(x, \alpha, t)$, α is a parameter, the x and t variables are separated by the assumption

So

$$\frac{\partial S}{\partial x} = \left(\frac{\partial W}{\partial x} \right) T \quad \dots(82)$$

$$\frac{\partial S}{\partial t} = \left(\frac{\partial T}{\partial t} \right) W \quad \dots(83)$$

The Eq. (80) becomes

$$\frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 e^{-\gamma t} T^2 + gx e^{\gamma t} + \frac{\partial T}{\partial t} W = 0 \quad \frac{\partial S}{\partial t} = \left(\frac{\partial T}{\partial t} \right) \quad \dots(84)$$

Divide Eq. (84) to $T^2 W$ and multiply by $e^{\gamma t}$

$$\frac{1}{2W} \left(\frac{\partial W}{\partial x} \right)^2 + \frac{e^{2\gamma t}}{T^2 W} gx + e^{\gamma t} \frac{\partial T}{\partial t} \frac{1}{T^2} = 0 \quad \dots(85)$$

Let

$$T = e^{\gamma t} \quad \dots(86)$$

So

$$\frac{\partial T}{\partial t} = \gamma e^{\gamma t} \quad \dots(87)$$

Substituting Eqs. (86) and (87) into Eq. (85), we get

$$\frac{1}{2W} \left(\frac{\partial W}{\partial x} \right)^2 + \frac{1}{W} gx + \gamma = 0 \quad \dots(88)$$

Eq. (88) becomes

$$\left(\frac{\partial W}{\partial x} \right)^2 = -2W \left(\frac{1}{W} gx + \gamma \right) \quad \dots(89)$$

We can write Eq. (89) as below

$$W'^2 + 2\gamma W + 2gx = 0 \quad \dots(90)$$

Differentiation Eq. (90) and W' replaced by y Eq. (90) becomes

$$2W' W'' + 2W' \gamma + 2g = 0 \quad \dots(91)$$

$$2yy' + 2\gamma y + 2g = 0 \quad \dots(92)$$

Divide Eq. (91) on 2

$$yy' + \gamma y + g = 0 \quad \dots(93)$$

Which is separable

$$y \frac{dy}{dx} = -(\gamma y + g) \quad \dots(94)$$

Integration Eq. (94), we get

$$\int \frac{\gamma dy}{-(\gamma y + g)} = \int dx \quad \dots(95)$$

Let

$$u = \gamma y + g \quad \dots(96)$$

So

$$y = \frac{u-g}{\gamma} \quad \dots(97)$$

Differentiate Eq. (97)

$$dy = \frac{du}{\gamma} \quad \dots(98)$$

Substituting Eqs. (97) and (98) into Eq. (95), we get

$$\int \frac{u-g}{(-u)\gamma^2} du = \int dx \quad \dots(99)$$

$$-\int \left(\frac{u}{u} - \frac{g}{u} \right) du = \gamma^2(x + c) \quad \dots(100)$$

Eq. (100) becomes

$$-u + g \ln(u) = \gamma^2(x + c) \quad \dots(101)$$

Instead of its value u in Eq. (101)

$$-\gamma y - g + g \ln(\gamma y + g) = \gamma^2(x + c) \quad \dots(102)$$

Differentiating Eq. (102) with respect to c

$$-\gamma y' - \frac{g\gamma y'}{\gamma y + g} = \gamma^2 \quad \dots(103)$$

I Unification of the denominators yield in Eq. (103), we have

$$\frac{-\gamma y' (\gamma y + g) - g\gamma y'}{(\gamma y + g)} = \gamma^2 \quad \dots(104)$$

$$-\gamma y' (\gamma y + g) - g\gamma y' = \gamma^2 (\gamma y + g) \quad \dots(105)$$

Take out (yy') a common factor in Eq. (105), we have

$$-\gamma y' (\gamma y) = \gamma^2 (\gamma y + g) \quad \dots(106)$$

Eq. (106) becomes

$$-\gamma y' = \gamma y + g \quad \dots(108)$$

So

$$-\frac{1}{2} 2\gamma y' = \gamma y + g \quad \dots(109)$$

Eq. (109) becomes

$$-\frac{1}{2} \frac{\partial y^2}{\partial c} = \gamma y + g \quad \dots(110)$$

I From Eq. (110)

$$W'^2 = -2\gamma W - 2gx \quad \dots(111)$$

Replace W' by y in Eq. (111), we obtain

$$y^2 = -2\gamma W - 2gx \quad \dots(112)$$

Note that when derived Eq. (112) the limit (-2gx) equal zero

$$\frac{\partial y^2}{\partial c} = -2\gamma \frac{\partial W}{\partial c} \quad \dots(113)$$

Multiply Eq. (113) by (-1/2), we get

$$-\frac{1}{2} \frac{\partial y^2}{\partial c} = \gamma \frac{\partial W}{\partial c} \quad \dots(114)$$

From Eqs. (110) and (114), we have

$$-\frac{1}{2} \frac{\partial y^2}{\partial c} = \gamma \frac{\partial W}{\partial c} = \gamma y + g \quad \dots(115)$$

From Eqs. (106) and (115), we get

$$\gamma \frac{\partial W}{\partial c} - g \ln \left(\gamma \frac{\partial W}{\partial c} \right) = -\gamma^2(x + c) \quad \dots(116)$$

Substituting Eq. (86) into Eq. (81)

$$S(x, \alpha, t) = W(x, \alpha) e^{\alpha t} \quad \dots(117)$$

I Then, if the parameter α is identified as c,

$$\beta = \frac{\partial S}{\partial \alpha} = e^{\alpha t} \frac{\partial S}{\partial c} \quad \dots(118)$$

So

$$\frac{\partial W}{\partial c} = \beta e^{-\alpha t} \quad \dots(119)$$

From Eq. (116), we obtain

$$x = \frac{1}{\gamma} \frac{\partial W}{\partial c} - g \ln \left(\frac{1}{\gamma} \frac{\partial W}{\partial c} \right) - c \quad \dots(120)$$

Substituting Eq. (119) into Eq. (120)

$$x = \frac{g}{\gamma^2} \ln(\beta\gamma) - c - \frac{\beta}{\gamma} e^{-\alpha t} - \frac{g}{\gamma} t \quad \dots(121)$$

Conclusion

In this paper, we have identified explicit time-dependent first integrals for the damped systems valid in different parameter regimes using the modified Hamilton-Jacobi approach. We have constructed the appropriate Hamiltonians from the time-dependent first integrals and transformed the corresponding Hamiltonian forms to standard Hamiltonian forms using suitable canonical transformations.

In addition, the solution of the Hamilton-Jacobi equations for such dissipative Hamiltonians have been constructed. We have derived an expression for the Hamilton-Jacobi equation and have applied our results for a number of time-dependent models including dissipation terms. Among them are:

friction linear in the velocity; friction quadratic in the velocity; friction quadratic in the velocity in a constant gravitational field; the linearly damped particle with constant force.

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Conflict of Interest

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References

- Bateman, H. (1931). On dissipative systems and related variational principles, *Phys. Rev.* 38, 815.
- Caldirola, P. (1983). Quantum Theory of Nonconservative Systems, *Nuovo Cimento* 18, 393 (1941); *ibid* 77, 241.
- Kanai, E. (1948). On the Quantization of Dissipative Systems, *Theor. Phys.* 3, 440.
- Havas, P. (1957). The Range of Application of the Lagrange Formalism, *Nuovo Cimento, Suppl.* 5, 363.
- Bates, Sand Weinstein, A. (1997). "Lectures on the geometry of quantization", 1st edition, American Mathematical Society, Berkeley Center for Pure and Applied Mathematics, Unites States of America.
- Goldstein, H. (1980). "Classical Mechanics", 2nd edition, Addison-Wesley. New York. USA.
- Sakurai, J, J. (1985). "Modern Quantum Mechanics", 1st edition Benjamin- Cummings Publishing Compong, Addison Wesley Longman, Unites States of America.
- Rabei, E, M. and Guler, Y. (1992). "Hamilton-Jacobi Treatment of Second-Class Constraints". *Physical Review A*, 46(6), 3513–3515.
- Rabei, M. (1996). "On Hamiltonian Systems, with Constraints", *Hadronic Journal*, 19, 597–605.
- Nawafleh, I, K. (2006). "Hamilton-Jacobi Formulation of the Reduced Phase Space", *Mu'tah Lil-Buhuth Wad-Dirasat: Natural and Applied Sciences Series* 21 (1), 11-15.
- Nawafleh, I, K. (2007). "Hamilton-Jacobi Formulation of Systems with Gauge Conditions", *Dirasat, Pure Science.* 34 (1), 119-124.
- Fowles, G. (1993). "Analytical Mechanics", 5th edition, Harcourt Brace and Company, United States of America.
- Batchelor, G, K. (1967). "An Introduction to Fluid Dynamics", 1st edition, Cambridge University Press, London.
- Campos, I. Jimenez, J. Del Valle, G. (2005). "A canonical Treatment of some systems with Friction", *European Journal of Physics*, V. 26, 711-725.
- Saletan, J, E. and Cromer, H, A. (1971). "Theoretical mechanics", 1st edition, Wiley, New York.
- Denman, H, H. and Buch, L, H. (1973). "Journal of Mathematical Physics", V.14
- Razavy, M. (2005). "Dissipative and Classical Quantum System", 1st edition, Imperial College Press, London.