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On Modification of Some Ratio Estimators using Parameters of Auxiliary Variable for the Estimation of the Population Mean

GERALD IKECHUKWU ONWUKA[1], WASIU AFOLABI BABAYEMI[1], AHMED AUDU[2] and AWWAL ADEJUMOBI[1]*

[1]Department of Mathematics, Faculty of Physical Sciences, Kebbi State University of Science and Technology, Aliero, Nigeria.
[2]Department of Statistics, Faculty of Science, Usmanu Danfodiyo University, Sokoto, Nigeria.

Abstract
Some existing estimators based on auxiliary attribute have been proposed by many authors. In this paper, we use the concept of power transformation to modify some existing estimators in order to obtain estimators that are applicable when there is positive or negative correlation between the study and auxiliary variable. The properties (Biases and MSEs) of the proposed estimators were derived up to the first order of approximation using Taylor series approach. The efficiency comparison of the proposed estimators over some existing estimators considered in the study were established. The empirical studies were conducted using existing population parameters to investigate the proficiency of the proposed estimators over some existing estimators. The results revealed that the proposed estimators have minimum Mean Square Errors and higher Percentage Relative Efficiencies than the conventional and other competing estimators in the study. These implies that the proposed estimators are more efficient and can produce better estimates of the population mean compared to the existing estimators considered in the study.

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Introduction

In sample surveys, auxiliary attribute is always used to increase the precision of estimated of population parameters. This can be done at either estimation or selection stage or both stages. The commonly used estimators, which make use of auxiliary variables, include ratio estimator, product estimator, regression and difference estimator. The classical ratio estimator is preferred when there is a high positive correlation between the variable of interest, Y and the auxiliary variable, X with the regression line passing through the origin. The classical product estimator, on the other hand is most preferred when there is a high negative correlation between Y and

CONTACT Awwal Adejumobi awwaladejumobi@gmail.com Department of Mathematics, Faculty of Physical Sciences, Kebbi State University of Science and Technology, Aliero, Nigeria.

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X while the linear regression estimator is most preferred when there is a high positive correlation between the two variables and the regression line of the study variable on the auxiliary has intercept on Y axis. The classical ratio and product estimators even though considered to be more useful in many practical situations have efficiencies which does not exceed that of the linear regression.

The use of auxiliary information has become indispensable for improving the exact of the estimators of population parameters like the mean and variance of the variable under study. A great variety of the techniques such as the ratio, product and regression methods of estimation are commonly known in this esteem. Keeping this fact in view, large number of estimators have been suggested in sampling literature. Some noteworthy contribution in this direction have been made by[1,2,4,5,6,7,8,9,11,12,13,15,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34] and many others.

The weaknesses discovered in this research is that[21] estimators are ratio-based, therefore they are only efficient when the correlation between study and auxiliary variables is positive. The efficiency of the estimator by[33] reduces as approaches zero in the presence of negative correlation between study and auxiliary variables.

To address the weaknesses in the[21,33] estimators, the estimators were modified using power transformation technique so as to obtain estimators that are applicable when the correlation between the study and auxiliary variables is either positive or negative. This study focuses on the modification of some ratio-based estimators using power transformation under simple random sampling in the presence of auxiliary variables and limited to the work of.[21,33]

Methodology

Let *U* denotes a finite population consisting of N units $\{U_1, U_2, \ldots, U_N\}$. Also, let (Y, X) denote the study variable and auxiliary variable taking values (y_i, x_i), (i = 1,2,.........., N), respectively, on the ith unit U_i of the population U. On the assumption that the population mean ($\bar{X}$) of X is known, the estimate of the population mean ($\bar{Y}$) of Y is obtained by selecting a sample of size n ($n<N$) from the population U using Simple Random Sampling without Replacement (SRSWOR) scheme.

N: Population size, n: Samplesize being selected from the entire population,

$f = n/N$: is the sampling fraction, $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$:

The population mean of the study variable Y,

$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$: The population mean of the auxiliaryvariable X.

$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$: The sample mean of the study variableY.

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$: The sample mean of the auxiliary variable X.

$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(X_i - \bar{X}\right)^2$: The finite population variance of the auxiliary variableX.

$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i - \bar{Y}\right)^2$: The finite population variance of the study variable Y.

$S_{yx} = \frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i - \bar{Y}\right)\left(X_i - \bar{X}\right)$: The finite population covariance between Y and X.

$C_y = \frac{S_y}{\bar{Y}}$: The population coefficients of variation of Y.

$C_x = \frac{S_x}{\bar{X}}$: The population coefficients of variation of X.

$\rho_{yx} = \frac{S_{yx}}{S_x S_y}$: Pearson's moment correlation coefficient of X and Y.

Q_i : The population ith Quartile of auxiliary variable.

$$\beta_1 = \frac{N\sum_{i=1}^{N}\left(X_i - \bar{X}\right)^3}{(N-1)(N-2)S_x^3}$$

: The population coefficient of skewness of auxiliary variable X.

$$\beta_2 = \frac{N(N+1)\sum_{i=1}^{N}\left(X_i - \bar{X}\right)^4}{(N-1)(N-2)(N-3)S_x^4} - \frac{3(N-1)^2}{(N-2)(N-3)}$$

: The population coefficient of kurtosis of auxiliary variable X.

$$QD = \frac{Q_3 - Q_1}{2}$$

: The population quartile deviation of auxiliary variable X.

Q_2N: The population second quartile of auxiliary variable X.

$$DM = \frac{D_1 + D_2 + \ldots + D_9}{9}$$

: The declile mean for auxiliary variable X.

ξ_0, ξ_1 : Error terms of the study and auxiliary variable.

Review of existing estimators

The conventional unbiased sample mean estimator is given by

$$\eta_0 = n^{-1}\sum_{i=1}^{n} y_i \quad ...(1)$$

The variance of $\bar{y}$ under SRSWOR sampling scheme is given by:

$$Var(\eta_0) = \psi_{n,N}\bar{Y}^2 C_y^2 \quad ...(2)$$

where $\psi_{n,N} = \left(\frac{1}{n} - \frac{1}{N}\right)$, $\bar{Y} = N^{-1}\sum_{i=1}^{N} y_i$

,

[8]Proposed conventional ratio estimator for the estimation of the population mean $\bar{Y}$ of the study variable, under the assumption that there is strong positive correlation between the study variable Y and auxiliary variable X. The proposed estimator is given by:

$$\eta_1 = \bar{y}\frac{\bar{X}}{\bar{x}} \quad ...(3)$$

The bias and MSE respectively of this estimator is given by

$$Bias(\eta_1) = \psi_{n,N}\bar{Y}\left(C_x^2 - \rho_{yx}C_yC_x\right)$$

$$MSE(\eta_1) = \psi_{n,N}\bar{Y}^2\left(C_y^2 + C_x^2 - 2\rho_{yx}C_yC_x\right) \quad ...(4)$$

[5]Suggested the following exponential type ratio and product estimators for estimation of the population mean $\bar{Y}$ as:

$$\eta_2 = \bar{y}\exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right) \quad ...(5)$$

The MSEs of the estimators are given by:

$$MSE(\eta_2) = \psi_{n,N}\bar{Y}^2\left[C_y^2 + \frac{C_x^2}{4} - \rho_{yx}C_yC_x\right] \quad ...(6)$$

[1]Proposed improved Ratio Estimator for the population mean using non-conventional measures of dispersion. The estimators are as follows:

$$\eta_3 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{(\bar{x} + G)}(\bar{X} + G) \quad ...(7)$$

$$\eta_4 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{(\bar{x}\rho + G)}(\bar{X}\rho + G) \quad ...(8)$$

$$\eta_5 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{(\bar{x}C_x + G)}(\bar{X}C_x + G) \quad ...(9)$$

$$\eta_6 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{(\bar{x} + D)}(\bar{X} + D) \quad ...(10)$$

$$\eta_7 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{(\bar{x}\rho + D)}(\bar{X}\rho + D) \quad ...(11)$$

$$\eta_8 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{(\bar{x}C_x + D)}(\bar{X}C_x + D) \quad ...(12)$$

$$\eta_9 = \frac{\bar{y} + b(\bar{X} - \bar{x})}{(\bar{x} + S_{pw})}(\bar{X} + S_{pw}) \quad ...(13)$$

$$\eta_{10} = \frac{\bar{y} + b(\bar{X} - \bar{x})}{(\bar{x}\rho + S_{pw})}(\bar{X}\rho + S_{pw}) \quad ...(14)$$

$$\eta_{11} = \frac{\bar{y} + b(\bar{X} - \bar{x})}{(\bar{x}C_x + S_{pw})}(\bar{X}C_x + S_{pw}) \quad ...(15)$$

The biases, related constants and the MSEs of the estimators are given by

$$B(\eta_3) = \frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p1}^2,\ R_{p1} = \frac{\bar{Y}}{(\bar{X}+G)},\ MSE(\eta_3) = \frac{(1-f)}{n}\left(R_{p1}^2S_x^2 + S_y^2(1-\rho^2)\right) \quad ...(16)$$

$$B(\eta_4) = \frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p2}^2,\ R_{p2} = \frac{\bar{Y}\rho}{(\bar{X}\rho+G)},\ MSE(\eta_4) = \frac{(1-f)}{n}\left(R_{p2}^2S_x^2 + S_y^2(1-\rho^2)\right) \quad ...(17)$$

$$B(\eta_5) = \frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p3}^2,\ R_{p3} = \frac{\bar{Y}C_x}{(\bar{X}C_x+G)},\ MSE(\eta_5) = \frac{(1-f)}{n}\left(R_{p3}^2S_x^2 + S_y^2(1-\rho^2)\right) \quad ...(18)$$

$$B(\eta_6) = \frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p4}^2,\ R_{p4} = \frac{\bar{Y}}{(\bar{X}+D)},\ MSE(\eta_6) = \frac{(1-f)}{n}\left(R_{p4}^2S_x^2 + S_y^2(1-\rho^2)\right) \quad ...(19)$$

$$B(\eta_7) = \frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p5}^2,\ R_{p4} = \frac{\bar{Y}\rho}{(\bar{X}\rho+D)},\ MSE(\eta_7) = \frac{(1-f)}{n}\left(R_{p5}^2S_x^2 + S_y^2(1-\rho^2)\right) \quad ...(20)$$

$$B(\eta_8) = \frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p6}^2,\ R_{p6} = \frac{\bar{Y}C_x}{(\bar{X}C_x+D)},\ MSE(\eta_8) = \frac{(1-f)}{n}\left(R_{p6}^2S_x^2 + S_y^2(1-\rho^2)\right) \quad ...(21)$$

$$B(\eta_9) = \frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p7}^2,\ R_{p7} = \frac{\bar{Y}}{(\bar{X}+S_{pw})},\ MSE(\eta_9) = \frac{(1-f)}{n}\left(R_{p7}^2S_x^2 + S_y^2(1-\rho^2)\right) \quad ...(22)$$

$$B(\eta_{10}) = \frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p8}^2,\ R_{p8} = \frac{\bar{Y}\rho}{(\bar{X}\rho+S_{pw})},\ MSE(\eta_{10}) = \frac{(1-f)}{n}\left(R_{p8}^2S_x^2 + S_y^2(1-\rho^2)\right) \quad ...(23)$$

$$B(\eta_{11}) = \frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p9}^2,\ R_{p9} = \frac{\bar{Y}C_x}{(\bar{X}C_x+S_{pw})},\ MSE(\eta_{11}) = \frac{(1-f)}{n}\left(R_{p9}^2S_x^2 + S_y^2(1-\rho^2)\right) \quad ...(24)$$

[21]Modified ratio estimator of population Mean using quartile and skewness coefficient. The estimators are as follows:

$$\eta_{12}=\frac{\bar{y}_n+b\left(\bar{X}_N-\bar{x}_n\right)}{\left(\bar{x}_n\beta_1+\varphi_1\right)}\left(\bar{X}_N\beta_1+\varphi_1\right) \quad ...(25)$$

$$\eta_{13}=\frac{\bar{y}_n+b\left(\bar{X}_N-\bar{x}_n\right)}{\left(\bar{x}_n\beta_1+\varphi_2\right)}\left(\bar{X}_N\beta_1+\varphi_2\right) \quad ...(26)$$

where, $\varphi_1 = (DM_N * Q_{2N})$ and $\varphi_2 = (DM_N * qd_N)$

The biases and the MSEs of the estimators are given by

$$B(\eta_{12})=\frac{1-f}{n}R_{r1}^2\frac{S_x^2}{\bar{Y}_N} \quad ...(27)$$

$$B(\eta_{13})=\frac{1-f}{n}R_{r2}^2\frac{S_x^2}{\bar{Y}_N} \quad ...(28)$$

$$MSE(\eta_{12})=\frac{1-f}{n}\left(\left(\frac{\bar{Y}_N\beta_1}{\left(X_N\beta_1+\varphi_1\right)}\right)^2 s_x^2+s_y^2\left(1-\rho_{yx}^2\right)\right) \quad ...(29)$$

$$MSE(\eta_{13})=\frac{1-f}{n}\left(\left(\frac{\bar{Y}_N\beta_1}{\left(X_N\beta_1+\varphi_2\right)}\right)^2 s_x^2+s_y^2\left(1-\rho_{yx}^2\right)\right) \quad ...(30)$$

where $R_{r1}=\frac{\bar{Y}_N\beta_1}{\left(X_N\beta_1+\varphi_1\right)}$, $R_{r2}=\frac{\bar{Y}_N\beta_1}{\left(X_N\beta_1+\varphi_2\right)}$

[33]Proposed a new alternative estimator by combining the ratio, product and exponential ratio type estimators using linear combination. The estimator is given as:

$$\eta_{14}=\bar{y}\left[k\frac{\bar{X}}{\bar{x}}+(1-k)\frac{\bar{x}}{\bar{X}}\right]\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right] \quad ...(31)$$

where, k = (0,1) is a suitably chosen constant to be determined.

The Bias and MSE of the proposed estimator are given by:

$$Bias(\eta_{14})=\bar{Y}\left[\frac{(16k-1)}{8}\lambda C_x^2+\frac{(1-4k)}{2}\lambda\rho_{(y,x)}C_yC_x\right]$$

$$MSE_{\min}(\eta_{14})=\psi_{n,N}\bar{Y}^2C_y^2\left(1-\rho_{(y,x)}^2\right) \quad ...(32)$$

where, the optimum value of k is

$$k_{(opt)}=\frac{2\rho_{(y,x)}C_y+C_x}{4C_x}.$$

The Proposed Estimators

Having studied the estimators of[21,33] and identified some weaknesses, the following proposed exponential-type estimators for estimating population mean $\bar{Y}$ under Simple Random Sampling without Replacement (SRSWOR) were suggested based on the motivation from the works of.[4,34] The proposed estimators are as given in (33) and (34).

$$\eta_1^{(*)}=\left(\bar{y}+b_\phi\left(\bar{X}-\bar{x}\right)\right)\left(\frac{\bar{x}}{\bar{X}}\right)^{\kappa_0}\exp\left(\frac{\left(\bar{X}\beta_1+\varphi_1\right)-\left(\bar{x}\beta_1+\varphi_1\right)}{\left(\bar{X}\beta_1+\varphi_1\right)+\left(\bar{x}\beta_1+\varphi_1\right)}\right) \quad ...(33)$$

$$\eta_2^{(*)}=\left(\bar{y}+b_\phi\left(\bar{X}-\bar{x}\right)\right)\left(\frac{\bar{x}}{\bar{X}}\right)^{\omega_0}\exp\left(\frac{\left(\bar{X}\beta_1+\varphi_2\right)-\left(\bar{x}\beta_1+\varphi_2\right)}{\left(\bar{X}\beta_1+\varphi_2\right)+\left(\bar{x}\beta_1+\varphi_2\right)}\right) \quad ...(34)$$

where, $\varphi_1=\left(DM_N\times Q_{2N}\right)$ and $\varphi_2=\left(DM_N\times qd_N\right)$,

$\kappa_0,\omega_0,\theta_1,\theta_2,b_\phi=\frac{S_{xy}}{S_x^2}$ are real constants.

Properties of the Proposed Estimators

In this section, the bias and MSE of the estimator proposed in this paper are derived and discussed.

Let, $\bar{y}_r=\bar{Y}\left(1+\xi_0\right),\ \bar{x}_r=\bar{X}\left(1+\xi_1\right),\ \forall\ |\xi_0|\approx 0, |\xi_1|\approx 0$

the first and second moment of ξ_1, *i*= 1,2 is

$$\left.\begin{aligned}&E(\xi_0)=E(\xi_1)=0,\ E(\xi_0^2)=\psi_{n,N}C_y^2, E(\xi_1^2)=\psi_{n,N}C_x^2\\&E(\xi_0\xi_1)=\psi_{n,N}\rho_{yx}C_yC_x,\quad \psi_{n,N}=\frac{1}{n}-\frac{1}{N}\end{aligned}\right\} \quad ...(35)$$

Theorem 1.1: To $O(n^{-1})$, bias of the proposed estimators $n_1^{(*)}$ is:

$$Bias\left(\eta_1^{(*)}\right)=\psi_{n,N}\left(\left(\frac{3}{8}\bar{Y}\theta^2-\frac{\bar{Y}\kappa_0\theta}{2}+\frac{b_\phi\bar{X}\theta}{2}-\kappa_0b_\phi\bar{X}+\frac{\bar{Y}\kappa_0\left(\kappa_0-1\right)}{2}\right)C_x^2+\left(\bar{Y}\kappa_0-\frac{\bar{Y}\theta}{2}\right)\rho_{yx}C_yC_x\right) \quad ...(36)$$

$$Bias\left(\eta_2^{(*)}\right)=\psi_{n,N}\left(\left(\frac{3}{8}\bar{Y}\gamma^2-\frac{\bar{Y}\omega_0\gamma}{2}+\frac{b_\phi\bar{X}\gamma}{2}-\omega_0b_\phi\bar{X}+\frac{\bar{Y}\omega_0\left(\omega_0-1\right)}{2}\right)C_x^2+\left(\bar{Y}\omega_0-\frac{\bar{Y}\gamma}{2}\right)\rho_{yx}C_yC_x\right) \quad ...(37)$$

Proof: Express (33) and (34) in terms of ξ_i, *i*=1,2, we have

$$\eta_1^{(*)} = \left(\bar{Y} + \bar{Y}\xi_0 - b_\phi \bar{X}\xi_1\right)\left(1 + \kappa_0\xi_1 + \frac{\kappa_0(\kappa_0 - 1)\xi_1^2}{2}\right)\exp\left(-\frac{\theta\xi_1}{2}\left(1 + \frac{\theta\xi_1}{2}\right)^{-1}\right)$$

$$\eta_1^{(*)} = \left(\bar{Y} + \bar{Y}\kappa_0\xi_1 + \frac{\bar{Y}\kappa_0(\kappa_0 - 1)\xi_1^2}{2} + \bar{Y}\xi_0 + \bar{Y}\kappa_0\xi_0\xi_1 - b_\phi\bar{X}\xi_1 - \kappa_0 b_\phi \bar{X}\xi_1^2\right)\exp\left(-\frac{\theta\xi_1}{2} + \frac{\theta^2\xi_1^2}{4}\right) \quad ...(38)$$

$$\eta_2^{(*)} = \left(\bar{Y} + \bar{Y}\xi_0 - b_\phi \bar{X}\xi_1\right)\left(1 + \omega_0\xi_1 + \frac{\omega_0(\omega_0 - 1)\xi_1^2}{2}\right)\exp\left(-\frac{\gamma\xi_1}{2}\left(1 + \frac{\gamma\xi_1}{2}\right)^{-1}\right)$$

$$\eta_2^{(*)} = \left(\bar{Y} + \bar{Y}\omega_0\xi_1 + \frac{\bar{Y}\omega_0(\omega_0 - 1)\xi_1^2}{2} + \bar{Y}\xi_0 + \bar{Y}\omega_0\xi_0\xi_1 - b_\phi\bar{X}\xi_1 - \omega_0 b_\phi \bar{X}\xi_1^2\right)\exp\left(-\frac{\gamma\xi_1}{2} + \frac{\gamma^2\xi_1^2}{4}\right) \quad ...(39)$$

where $\theta = \left(\frac{\bar{X}\beta_1}{\bar{X}\beta_1 + \varphi_1}\right)$, $\gamma = \left(\frac{\bar{X}\beta_1}{\bar{X}\beta_1 + \varphi_2}\right)$

Simplifying (38) and (39) up $O(n^{-1})$, we have

$$\eta_1^{(*)} = \bar{Y} + \bar{Y}\xi_0 + \left(\bar{Y}\kappa_0 - \frac{\bar{Y}\theta}{2} - b_\phi\bar{X}\right)\xi_1 + \left(\frac{3}{8}\bar{Y}\theta^2 - \frac{\bar{Y}\kappa_0\theta}{2} + \frac{b_\phi\bar{X}\theta}{2} - \kappa_0 b_\phi\bar{X} + \frac{\bar{Y}\kappa_0(\kappa_0 - 1)}{2}\right)\xi_1^2 + \left(\bar{Y}\kappa_0 - \frac{\bar{Y}\theta}{2}\right)\xi_0\xi_1 \quad ...(40)$$

$$\eta_2^{(*)} = \left(\bar{Y} + \bar{Y}\xi_0 + \left(\bar{Y}\omega_0 - \frac{\bar{Y}\gamma}{2} - b_\phi\bar{X}\right)\xi_1 + \left(\frac{3}{8}\bar{Y}\gamma^2 - \frac{\bar{Y}\omega_0\gamma}{2} + \frac{b_\phi\bar{X}\gamma}{2} - \omega_0 b_\phi\bar{X} + \frac{\bar{Y}\omega_0(\omega_0 - 1)}{2}\right)\xi_1^2 + \left(\bar{Y}\omega_0 - \frac{\bar{Y}\gamma}{2}\right)\xi_0\xi_1\right) \quad ...(41)$$

Subtract $\bar{Y}$ from both sides of (40) and (41), take expectation and apply the results of (35), theorem 1.1 is proved.

Theorem 1.2: To $O(n^{-1})$, MSE of the proposed estimators $n_1^{(*)}$ is:

$$MSE_{\min}\left(\eta_1^{(*)}\right) = \psi_{n,N}\bar{Y}^2\left(C_y^2\left(1 - \rho_{yx}^2\right) + \rho_{yx}C_yC_x(1 - \theta)\right) \quad ...(42)$$

$$MSE_{\min}\left(\eta_2^{(*)}\right) = \psi_{n,N}\bar{Y}^2\left(C_y^2\left(1 - \rho_{yx}^2\right) + \rho_{yx}C_yC_x(1 - \gamma)\right) \quad ...(43)$$

Proof: Subtract $\bar{Y}$ from both sides of (40) and (41), we have:

$$\eta_1^{(*)} - \bar{Y} = \bar{Y}\xi_0 + \left(\bar{Y}\kappa_0 - \frac{\bar{Y}\theta}{2} - b_\phi\bar{X}\right)\xi_1 + \left(\frac{3}{8}\bar{Y}\theta^2 - \frac{\bar{Y}\kappa_0\theta}{2} + \frac{b_\phi\bar{X}\theta}{2} - \kappa_0 b_\phi\bar{X} + \frac{\bar{Y}\kappa_0(\kappa_0 - 1)}{2}\right)\xi_1^2 + \left(\bar{Y}\kappa_0 - \frac{\bar{Y}\theta}{2}\right)\xi_0\xi_1 \quad ...(44)$$

$$\eta_2^{(*)} - \bar{Y} = \left(\bar{Y}\xi_0 + \left(\bar{Y}\omega_0 - \frac{\bar{Y}\gamma}{2} - b_\phi\bar{X}\right)\xi_1 + \left(\frac{3}{8}\bar{Y}\gamma^2 - \frac{\bar{Y}\omega_0\gamma}{2} + \frac{b_\phi\bar{X}\gamma}{2} - \omega_0 b_\phi\bar{X} + \frac{\bar{Y}\omega_0(\omega_0 - 1)}{2}\right)\xi_1^2 + \left(\bar{Y}\omega_0 - \frac{\bar{Y}\gamma}{2}\right)\xi_0\xi_1\right) \quad ...(45)$$

Square both sides of (44) and (45) then simplify up to $O(n^{-1})$, we get

$$\left(\eta_1^{(*)} - \bar{Y}\right)^2 = \bar{Y}^2\xi_0^2 + \left(\bar{Y}\kappa_0 - \frac{\bar{Y}\theta}{2} - b_\phi\bar{X}\right)^2\xi_1^2 + 2\bar{Y}\left(\bar{Y}\kappa_0 - \frac{\bar{Y}\theta}{2} - b_\phi\bar{X}\right)\xi_0\xi_1 \quad ...(46)$$

$$\left(\eta_2^{(*)} - \bar{Y}\right)^2 = \bar{Y}^2\xi_0^2 + \left(\bar{Y}\omega_0 - \frac{\bar{Y}\gamma}{2} - b_\phi\bar{X}\right)^2\xi_1^2 + 2\bar{Y}\left(\bar{Y}\omega_0 - \frac{\bar{Y}\gamma}{2} - b_\phi\bar{X}\right)\xi_0\xi_1 \quad ...(47)$$

where $\kappa_0 = \frac{\theta}{2}$, $\omega_0 = \frac{\gamma}{2}$

Take expectation of (46), (47) and apply the results of (35), theorem 1.2 is proved.

Efficiency Comparison

In this section, conditions for the efficiency of the new estimators over some existing related estimators established were established.

Theorem 1.3: Estimator $n_1^{(*)}$ is more efficient than n_1 if (48) and (49) is satisfied.

$$\left|\rho_{yx}\right| < \frac{(1 - \theta)C_x}{C_y} \quad ...(48)$$

$$\left|\rho_{yx}\right| < \frac{(1 - \gamma)C_x}{C_y} \quad ...(49)$$

Proof: Minus (42) and (43) from (2), theorem 1.3 is proved.

Theorem 1.4: Estimator $n_1^{(*)}$ is more efficient than n_1 if (50) and (51) is satisfied.

$$\left|\rho_{yx}\right| < \frac{C_x\left((2 - \theta)C_y^2 + \sqrt{(2 - \theta)^2 C_y^2 - 1}\right)}{2} \quad ...(50)$$

$$\left|\rho_{yx}\right| < \frac{C_x\left((2 - \gamma)C_y^2 + \sqrt{(2 - \gamma)^2 C_y^2 - 1}\right)}{2} \quad ...(51)$$

Proof: Minus (42) and (43) from (4), theorem 1.4 is proved.

Theorem 1.5: Estimator $n_1^{(*)}$ is more efficient than n_1 if (52) and (53) is satisfied.

$$|\rho_{yx}| < \frac{\left(\frac{\beta_1}{(\bar{X}_N\beta_1+\varphi_1)}\right)^2}{(1-\theta)(C_yC_x)}, \quad |\rho_{yx}| < \frac{\left(\frac{\beta_1}{(\bar{X}_N\beta_1+\varphi_2)}\right)^2}{(1-\theta)(C_yC_x)} \qquad ...(52)$$

$$|\rho_{yx}| < \frac{\left(\frac{\beta_1}{(\bar{X}_N\beta_1+\varphi_1)}\right)^2}{(1-\gamma)(C_yC_x)}, \quad |\rho_{yx}| < \frac{\left(\frac{\beta_1}{(\bar{X}_N\beta_1+\varphi_2)}\right)^2}{(1-\gamma)(C_yC_x)} \qquad ...(53)$$

Proof: Minus (42) and (43) from (29) and (30), theorem 1.5 is proved.

Theorem 1.6: Estimator $n_1^{(*)}$ is more efficient than n_1 (54) and (55) is satisfied.

$$|\rho_{yx}| < 0 \qquad ...(54)$$

$$|\rho_{yx}| < 0 \qquad ...(55)$$

Proof: Minus (42) and (43) from (32), theorem 1.6 is proved.

Test for the Consistency of the Modified Estimators

In this section, the consistencies of the modified estimators $n_1^{(*)}$, and $n_2^{(*)}$ were established.

Proof: Let f(x) and g(x) be continuous function, then

$$\lim_{n\to N}\left(f(x)\pm g(x)\right) = \lim_{n\to N} f(x) \pm \lim_{n\to N} g(x), \quad N \neq \infty \qquad ...(56)$$

$$\lim_{n\to N}\left(f(x)\times g(x)\right) = \lim_{n\to N} f(x) \times \lim_{n\to N} g(x), \quad N \neq \infty \qquad ...(57)$$

$$\lim_{n\to N}\frac{f(x)}{g(x)} = \frac{\lim_{n\to N} f(x)}{\lim_{n\to N} g(x)}, \quad N \neq \infty, \ \lim_{n\to N} g(x) \neq 0 \qquad ...(58)$$

As $n \to N$, $n = N$. Using the results of (56), (57) and (58), we have

$$\lim_{n\to N}\left(\eta_1^{(*)}\right) = \left(\lim_{n\to N}\bar{y} + b_\phi\left(\bar{X} - \lim_{n\to N}\bar{x}\right)\right)\left(\frac{\lim_{n\to N}\bar{x}}{\bar{X}}\right)^{\kappa_0} \exp\left(\frac{(\bar{X}\beta_1+\varphi_1)-\left(\beta_1\lim_{n\to N}\bar{x}+\varphi_1\right)}{(\bar{X}\beta_1+\varphi_1)+\left(\beta_1\lim_{n\to N}\bar{x}+\varphi_1\right)}\right) \qquad ...(59)$$

$$\lim_{n\to N}\left(\eta_2^{(*)}\right) = \left(\lim_{n\to N}\bar{y} + b_\phi\left(\bar{X} - \lim_{n\to N}\bar{x}\right)\right)\left(\frac{\lim_{n\to N}\bar{x}}{\bar{X}}\right)^{\omega_0} \exp\left(\frac{(\bar{X}\beta_2+\varphi_2)-\left(\beta_2\lim_{n\to N}\bar{x}+\varphi_2\right)}{(\bar{X}\beta_2+\varphi_2)+\left(\beta_2\lim_{n\to N}\bar{x}+\varphi_2\right)}\right) \qquad ...(60)$$

As $n \to N$, $\lim_{n\to N}\bar{y} = Y$, $\lim_{n\to N}\bar{x} = X$, Therefore,

$$\lim_{n\to N}\left(\eta_1^{(*)}\right) = \bar{Y} \quad \text{and} \quad \lim_{n\to N}\left(\eta_2^{(*)}\right) = \bar{Y} \qquad ...(61)$$

Hence, the estimators $n_1^{(*)}$, and $n_2^{(*)}$ are consistent.

Empirical Study

In this section, real life data was conducted to examine the superiority of the proposed estimators over the existing estimators considered in the study. Natural dataset, population 1, 2, 3 and 4 as used is given in.[32,14,11,16]

$N=150$, $n=40$, $\bar{Y}=79.58$, $\bar{X}=6.5833$, $\rho=0.9363$, $C_y=0.781333$, $C_x=0.661726$, $S_y=62.1785$, $S_x=4.3564$, $\beta_1=1.4984$, $\beta_2=5.408$, $Q_1=4$, $M_d=5$, $Q_3=10$, $QD=3$, $DM=6.22$, $TM=6$, $MR=11$, $HL=7$, $G=8.2298$, $D=9.2542$, $S_{pw}=9.3707$

Population 2

The data is defined as follows:

$N=80$, $n=20$, $\bar{Y}=5182.637$, $\bar{X}=1126.463$, $\rho=0.941$, $C_y=0.354$, $C_x=0.751$, $S_y=1835.659$, $S_x=845.610$, $\beta_1=1.050$, $\beta_2=0.063$, $M_d=757.500$, $QD=588.125$, $DM=1052.222$, $TM=931.562$, $MR=1795.5$, $HL=1040.5$, $G=8.2298$, $D=1295.00$, $S_{pw}=9.3707$

Population 3

The data is defined as follows:

$N=106$, $n=40$, $\bar{Y}=2212.59$, $\bar{X}=27421.70$, $\rho=0.860$, $C_y=5.22$, $C_x=2.10$, $S_y=11551.53$, $S_x=57460.61$, $\beta_1=2.122$, $\beta_2=34.572$, $M_d=7297.50$, $QD=12156.25$, $DM=1052.222$, $TM=931.562$, $MR=1795.5$, $HL=1040.5$, $G=40201.69$, $D=35634.99$, $S_{pw}=35298.81$

Population 4

The data is defined as follows:

$N=34$, $n=20$, $\bar{Y}=856.4117$, $\bar{X}=199.4412$, $\rho=0.4453$, $C_y=0.8561$, $C_x=0.7531$, $S_y=733.1407$, $S_x=150.2150$, $\beta_1=1.1823$, $\beta_2=1.0445$, $M_d=142.50$, $QD=80.25$, $DM=5.22$, $TM=931.562$, $MR=1795.5$, $HL=1040.5$, $G=162.996$, $D=144.481$, $S_{pw}=142.990$

Table 1 above show the numerical results of the Mean Square Errors (MSEs) of the estimators and using four natural data sets of all the subjects examined, the two proposal have a minimum MSE for all data sets. This implies that the proposed methods have shown a high level of efficiency on others considered in the study, and can produce better estimate of the population parameters than the existing estimators.

Table 2 above show the numerical results of the Percentage Relative Efficiencies (PREs) of the estimators and using four natural data sets of all the subjects examined, the two proposal have the highest PREs for all data sets. This implies that the proposed methods have shown a high level of efficiency on others considered in the study, and can produce better estimate of the population parameters than the existing estimators.

$$PRE = \frac{Var(\bar{y})}{MSE(\eta_i) \ \ or \ \ MSE_{\min}\left(\eta_i^{(*)}\right)} * 100$$

where $Var(\bar{y})$ is the variance of sample mean, $MSE_{\min}\left(\eta_i^{(*)}\right)$ is the mean square error values of the proposed estimator in section 3 and $MSE(\eta_i)$ is the mean square error values of the existing estimators mentioned in section 2.

Table 1: Mean Square Errors of the Proposed Estimators and Existing Estimators Using the Population 1,2,3,4

Estimators	Popn. 1	Popn. 2	Popn. 3	Popn. 4
η_0	70.87966	126223.3	2076448	11067.09
η_1	9.308712	190347.9	975702.7	10960.84
η_2	27.38418	16264.54	1442060	8872.9
η_3	18.7843	573863.1	595897.2	11465.62
η_4	18.06615	573354.7	586615.7	9937.329
η_5	14.83329	571182.4	656912.9	10841.85
η_6	17.52726	137304.5	604155.2	11752.41
η_7	16.86835	129463.4	593942.3	10113.18
η_8	13.95094	103090.5	668560.2	11097.2
η_9	17.39943	572739.9	604835.4	11777.55
η_{10}	16.74693	572162.8	594550	10129.2
η_{11}	13.86291	569698.2	669486.1	11119.89
η_{12}	11.6906	33593.75	822292.3	13699.37
η_{13}	14.82279	14472.88	540883.3	9982.922
η_{14}	9.109983	14455.17	540719.9	8872.762
$\eta_1(*)$	**8.742461**	**14452.96**	**540707.7**	**8872.571**
$\eta_2(*)$	**8.908933**	**14454.21**	**540715.7**	**8872.642**

Table 2: Percentage Relative Efficiencies of the Proposed Estimators and Existing Estimators Using the Population 1,2,3,4

Estimators	Popn. 1	Popn. 2	Popn. 3	Popn. 4
η_0	100	100	100	100
η_1	761.4336	66.31186	212.8156	100.9693
η_2	258.8343	776.0641	143.9918	124.7291
η_3	377.3346	21.99536	348.4573	96.52413
η_4	392.3342	22.01486	353.9707	111.3688
η_5	477.8418	22.09859	316.0918	102.0774
η_6	404.3967	91.92946	343.6944	94.16863
η_7	420.1932	97.49725	349.6043	109.4324
η_8	508.0637	122.4393	310.585	99.72868
η_9	407.3677	22.0385	343.3079	93.96762
η_{10}	423.2397	22.06072	349.2469	109.2593
η_{11}	511.29	22.15616	310.1554	99.52516
η_{12}	606.2963	375.7344	252.5194	80.78534
η_{13}	478.1802	872.1367	383.8994	110.8602
η_{14}	778.0438	873.2052	384. 0183	124.7211
$\eta_1(*)$	810.7518	873.2275	384.0246	124.7337
$\eta_2(*)$	804.6021	873.2125	384.0241	124.7325

Conclusion

By considering the results obtained from the empirical study on the efficiency of the suggested estimators over some exists related estimators considered in the study. From the empirical study, the results revealed that the suggested estimators $n_1^{(*)}$, and $n_2^{(*)}$ have minimum mean square error and higher percentage relative efficiency compared to other estimators considered in the numerical computations carried out in the study. In the other words, the suggested estimators $n_1^{(*)}$, and $n_2^{(*)}$ have higher chance of producing estimate that is closer to the true value of the population mean than other estimators considered in the literature of this study.

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Conflict of Interest

The authors declare no conflict of interest.

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